## SAMPLE SOLUTIONS EXERCISE 10

### EXERCISE 10.1: THE FIRST GAP **(6P)**

The Hamiltonian of the Ising quantum chain is defined by

$$\mathbf{H}_N = \sum_{n=0}^{N-1} \Big( \sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^z \Big)$$

where $\lambda > 0$ and $\sigma_n^{x|y|z} = \mathbb{1}_{2^n \times 2^n} \otimes \sigma^{x|y|z} \otimes \mathbb{1}_{2^{N-n-1} \times 2^{N-n-1}}$ are Pauli matrices acting at site $n$, assuming periodic boundary conditions. The first (or lowest) gap $\Delta_N$ is defined as the difference between the first two lowest-lying eigenvalues of $\mathbf{H}_N$.

(a) In Markov processes, the first gap of $\mathcal{L}$ is the inverse leading relaxation time. In particle physics, the first gap measures the mass. What is the interpretation of the first gap of $\mathbf{H}_N$ in the present context? (1P)

(b) With the *Mathematica*® code fragment for the tensor product $\otimes$ given in the lecture notes,[1] write a code snippet to set up the Hamiltonian for arbitrary $N$ and $\lambda$. (2P)
Cross check: The eigenvalues for $N = 3$ (i.e. 8×8) are $\big\{ -\lambda - 1, -\lambda - 1, \lambda - 1, \lambda - 1, -2\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1, 2\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1, -2\sqrt{\lambda^2 + \lambda + 1} - \lambda + 1, 2\sqrt{\lambda^2 + \lambda + 1} - \lambda + 1 \big\}$

(c) With (b) compute *numerically*[2] the first gap of $\mathbf{H}_N$ for $N = 2, 4, 6, 8, 10, 12$ and and plot the gap for $\lambda = 0.8, 1, 1.2$ double-logarithmically as a function of $N$. For which $\lambda$ do you get an almost straight line of points and why?[3] (2P)

(d) Compute the negative *discrete logarithmic derivative*

$$\nu(N) = -\frac{\ln[\Delta_{N+2}/\Delta_N]}{\ln[(N+2)/N]}$$

for $\lambda = 1$ and $N = 2, 4, 6, 8, 10$ and guess the value of the critical exponent $\nu$ which is defined as the limit $\nu = \lim_{N\to\infty} \nu(N)$. (1P)

### SAMPLE SOLUTION

(a) In the present case the Hamiltonian $\mathbf{H}$ is related to the transfer matrix $\mathbf{T} = e^{\mathbf{H}}$, hence the first gap can be interpreted as the *correlation length* in vertical direction. (1P)

(b) A possible solution reads:

```
 Attributes[CircleTimes] = {Flat, OneIdentity};
CircleTimes[a_List /; VectorQ[a], b_List /; VectorQ[b]] :=
  Flatten[KroneckerProduct[a, b]];
```

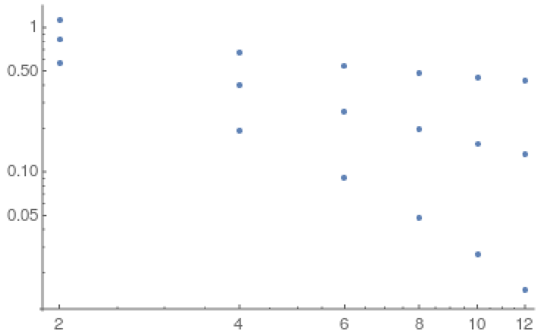
[1] You are of course free to use any other algebraic computer systems as well.
[2] In Mathmatica: Use `N[...]` to make **H** numerical: `Eigenvalues[N[H[length,lambda]]]`
[3] The execution for $N = 12$ should take less than 20 seconds. If not, go only up to $N = 10$.

```
CircleTimes[a_List /; MatrixQ[a], b_List /; MatrixQ[b]] :=
  KroneckerProduct[a, b];
  sx = {{0, 1}, {1, 0}};
sy = {{0, -I}, {I, 0}};
sz = {{1, 0}, {0, -1}};
Id[n_] := IdentityMatrix[n];

 H[L_, \[Lambda]_] := Sum[
   Id[2^j]\[CircleTimes]sx\[CircleTimes]sx\[CircleTimes]Id[2^(
     L - j - 2)], {j, 0, L - 2}] +
  sx\[CircleTimes]Id[2^(L - 2)]\[CircleTimes]sx +
  \[Lambda]*
   Sum[Id[2^j]\[CircleTimes]sz\[CircleTimes]Id[2^(L - j - 1)], {j, 0,
     L - 1}]
```

(c) To extract the gap you can use e.g. the following function:

```
 gap[L_, \[Lambda]_] := Module[{spec},
  spec = H[L, \[Lambda]] // N // Eigenvalues // Sort;
  spec[[2]] - spec[[1]]
  ]
```




Only the points for $\lambda = 1$ are almost straight. The reason is that $\lambda_c = 1$ is the critical point (see lecture notes), and at the critical point we expect power laws which give straight lines in double-logarithmic representations.

(d) The logarithmic derivative gives

$$\{1.05824, 1.01784, 1.00874, 1.0052, 1.00345\}$$

and we would conjecture that $\nu = 1$. If you wish to do it more carefully, you may e.g. plot these values against $1/N^2$, which gives again an almost straight line that you can extrapolate to this value. $\nu = 1$ is in fact the correct result for the Ising correlation length exponent in 2D (= 1D quantum).

## Exercise 10.2: Duality of the anisotropic $XY$ chain (6P)

Duality requires AdS/CFT? Not at all! There are much simpler toy models which exhibit a duality between bulk and boundary behavior. One such example is the anisotropic $XY$ quantum chain. The Hamiltonian on a 1D chain of $L$ sites with open boundaries is defined by

$$\mathbf{H}(\eta,q) = -\frac{1}{2}\sum_{j=1}^{L-1}\Big(\eta\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y + q\sigma_j^z + q^{-1}\sigma_{j+1}^z\Big),$$

where $\eta > 0$ controls the bulk anisotropy while $q > 0$ controls the boundary terms (since in the bulk the combination $\frac{1}{2}(q+q^{-1})\sigma^z$ prevails).

(a) Let us write $\mathbf{H}(\eta,q) = \sum_{j=1}^{L-1} e_j$, where $e_j$ is the 2-site Hamiltonian acting on sites $j$ and $j+1$. Show that the $e_j$ obey the algebra (2P)

$$e_j^2 = A\,\mathbb{1}_L\,, \quad \Big(e_je_{j+1}e_j - e_{j+1}e_je_{j+1} + (A-1)(e_j - e_{j+1})\Big)\Big(e_j - e_{j+1}\Big) = B\,\mathbb{1}_L\,,$$

where $A, B$ are constants which depend on $\eta, q$ while $\mathbb{1}_L$ is the $2^L \times 2^L$ unit matrix.

(b) Analyze how $A$ and $B$ depend on $\eta$ and $q$. Why is it justified to conjecture that there should exist a similarity transformation $\mathbf{U}$ which swaps these parameters in such a way that (1P)

$$\mathbf{H}(\eta,q) \;=\; \mathbf{U}\,\mathbf{H}(q,\eta)\mathbf{U}^{-1}\ ?$$

(c) The *Jordan Wigner transformation* is defined as

$$\tau_j^{x,y} \;=\; \Big(\prod_{i=1}^{j-1}\sigma_i^z\Big)\sigma_j^{x,y}.$$

The purpose of this transformation is that these operators anticommute also on different sites. Express the Hamiltonian in terms of $\tau_j^x$ and $\tau_j^y$. (2P)

(d) Show e.g. with *Mathematica®* that for a 3-site system the similarity transformation is of the form

$$\mathbf{U} = \mathbb{1} + \omega\sum_{1\le j<k\le 3}\tau_j^x\tau_k^x\,.$$

Determine the unknown constant $\omega$ as a function of $\eta$ and $q$. (1P)

**Remark:** Explicit expressions for $\mathbf{U}$ can be found for any system size, establishing an exact duality for all $\eta, q, L$.

## Sample Solution

(a) The interaction matrix $e_1$ acting on two sites reads

$$e_1 = \frac{1}{4}\begin{pmatrix} q+q^{-1} & 0 & 0 & \eta-\eta^{-1} \\ 0 & q-q^{-1} & \eta+\eta^{-1} & 0 \\ 0 & \eta+\eta^{-1} & q-q^{-1} & 0 \\ \eta-\eta^{-1} & 0 & 0 & -q-q^{-1}\end{pmatrix},$$

hence (1P)

$$\Rightarrow \quad e_1^2 = A\mathbb{1}_2 \quad \text{where} \quad A = \frac{1}{4}\left(\eta^2 + \frac{1}{\eta^2} + q^2 + \frac{1}{q^2}\right).$$

The same result for $A$ holds of course on chains with more than two sites. Now we can explore the second equation. This requires at least 3 sites ($8 \times 8$ matrices). We compute (e.g. with *Mathematica®* ) the matrix

$$\Big(e_1 e_2 e_1 - e_2 e_1 e_2 + (A-1)(e_1 - e_2)\Big)(e_1 - e_2)$$

which indeed turns out to be proportional to the identity matrix times (1P)

$$B = \frac{1}{4}(q - q^{-1})^2(\eta - \eta^{-1})^2.$$

It is clear that the same relation holds for $e_2, e_3$ and $e_3, e_4$ and so on, while non-neighboring matrices such as $e_1$ and $e_3$ commute.

(b) If we take a closer look at the constants $A$ and $B$ we realize that both constants are invariant under the exchange $\eta \leftrightarrow q$. Therefore, the entire algebra given in (a) is invariant under $\eta \leftrightarrow q$. Since this algebra play the role of a spectrum-generating algebra (analogous to the Temperley-Lieb algebra discussed earlier), it is clear that $\mathbf{H}(\eta, q)$ and $\mathbf{H}(q, \eta)$ have the same spectrum, suggesting that they are related by a similarity transformation. This similarity transformation would consist of a product of the respective transformations which diagonalize these Hamiltonians.[4]

(c) For products of two neighboring $\tau$'s the Jordan Wigner string $\prod_{i=1}^{j-1} \sigma_j^z$ cancels everywhere to the left of the product (due to $(\sigma^z)^2 = \mathbb{1}$) so that only one $\sigma^z$ survives between the two sites. Here is an example:

$$\tau_j^x \tau_{j+1}^y \;=\; \underbrace{(\sigma_1^z)^2}_{=\mathbb{1}} \cdots \underbrace{(\sigma_{j-1}^z)^2}_{=\mathbb{1}} \underbrace{\sigma_j^x \sigma_j^z}_{=-i\sigma_j^y} \sigma_{j+1}^y \;=\; -i\sigma_j^y \sigma_{j+1}^y.$$

Similarly other products can be evaluated. Altogether the results read

$$\sigma_j^x \sigma_{j+1}^x = -i\tau_j^y \tau_{j+1}^x\,, \quad \sigma_j^y \sigma_{j+1}^y = i\tau_j^x \tau_{j+1}^y\,, \quad \sigma_j^z = -i\tau_j^x \tau_j^y\,.$$

This allows us to rewrite the Hamiltonian as

$$\mathbf{H}(\eta, q) = \frac{i}{2}\sum_{j=1}^{L-1}\Big(\eta\tau_j^y \tau_{j+1}^x - \eta^{-1}\tau_j^x \tau_{j+1}^y + q\tau_j^x \tau_j^y + q^{-1}\tau_{j+1}^x \tau_{j+1}^y\Big)$$

(d) The suggested similarity transformation $\mathbf{U} = \mathbb{1} + \omega \sum_{1 \le j < k \le 3} \tau_j^x \tau_k^x$ corresponds to the matrix

$$\mathbf{U} \;=\; \begin{pmatrix} 1 & 0 & 0 & -\omega & 0 & -\omega & -\omega & 0 \\ 0 & 1 & -\omega & 0 & -\omega & 0 & 0 & -\omega \\ 0 & \omega & 1 & 0 & -\omega & 0 & 0 & \omega \\ \omega & 0 & 0 & 1 & 0 & -\omega & \omega & 0 \\ 0 & \omega & \omega & 0 & 1 & 0 & 0 & -\omega \\ \omega & 0 & 0 & \omega & 0 & 1 & -\omega & 0 \\ \omega & 0 & 0 & -\omega & 0 & \omega & 1 & 0 \\ 0 & \omega & -\omega & 0 & \omega & 0 & 0 & 1 \end{pmatrix}$$

[4] This is only a conjecture because it is not clear at this stage whether the two Hamiltonians have the same degeneracies.

while the 3-site Hamiltonian reads

$$\mathbf{H} = \frac{1}{2}\begin{pmatrix} 2\left(q+\frac{1}{q}\right) & 0 & 0 & \eta-\frac{1}{\eta} & 0 & 0 & \eta-\frac{1}{\eta} & 0 \\ 0 & 2q & \eta+\frac{1}{\eta} & 0 & 0 & 0 & 0 & \eta-\frac{1}{\eta} \\ 0 & \eta+\frac{1}{\eta} & 0 & 0 & \eta+\frac{1}{\eta} & 0 & 0 & 0 \\ \eta-\frac{1}{\eta} & 0 & 0 & -\frac{2}{q} & 0 & \eta+\frac{1}{\eta} & 0 & 0 \\ 0 & 0 & \eta+\frac{1}{\eta} & 0 & \frac{2}{q} & 0 & 0 & \eta-\frac{1}{\eta} \\ 0 & 0 & 0 & \eta+\frac{1}{\eta} & 0 & 0 & \eta+\frac{1}{\eta} & 0 \\ \eta-\frac{1}{\eta} & 0 & 0 & 0 & 0 & \eta+\frac{1}{\eta} & -2q & 0 \\ 0 & \eta-\frac{1}{\eta} & 0 & 0 & \eta-\frac{1}{\eta} & 0 & 0 & -\frac{2(q^2+1)}{q} \end{pmatrix}$$

With *Mathematica*® we can simply type

```
Solve[H[eta,q].U == U.H[q,eta], omega]
```

finding the result (1P)

$$\boxed{\omega = \frac{\eta - q}{\eta + q}.}$$

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**($\Sigma$ = 12P)**