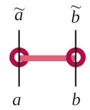
Sample Solutions Exercise 6

Exercise 6.1: Fast Fourier Transform as a tensor network (5P)



According to the lecture notes the fast Fourier transformation relies on a recursive bisection which can be represented graphically as in the figure shown above.

(a) Explain how the entangler (the red horizontal connection between two line) works, that is, determine $\tilde{a}$ and $\tilde{b}$ as functions of a and b. Make yourself familiar with elementary quantum gates in quantum computing. Which elementary quantum gate does the horizontal line correspond to? (2P)

(b) Let us consider the Fourier transformation $\mathcal{F}^{(4)}$ which acts on four complex numbers $\{a_0, a_1, a_2, a_3\}$. Apply the recursion graphically two times and draw the resulting tensor network (which contains only vertical lines, green phase shifts boxes, and horizontal connections but no red $\mathcal{F}$-boxes). Specify the individual phase shifts of the green boxes. (2P)

(c) At each solid line in the expanded circuit, specify the specific value running along the line in terms of a_0, a_1, a_2, a_3. In particular, express $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ in terms of a_0, a_1, a_2, a_3. Verify that the network indeed realizes a DFT for $N = 4$. (1P)

Sample Solution

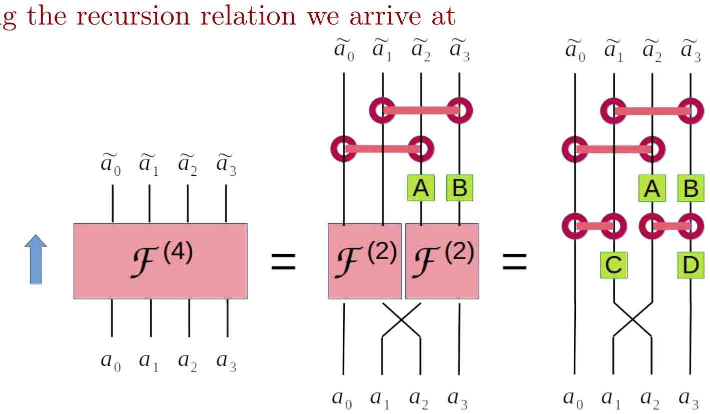
(a) The relation reads: (1P)

$$(\tilde{a}_0 = \frac{1}{\sqrt{2}} a_0 + a_1)\,, \qquad \tilde{a}_1 = \frac{1}{\sqrt{2}}(a_0 - a_1).$$

The horizontal line corresponds Hadamard gate: (1P)

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(b) Expanding the recursion relation we arrive at (2P)

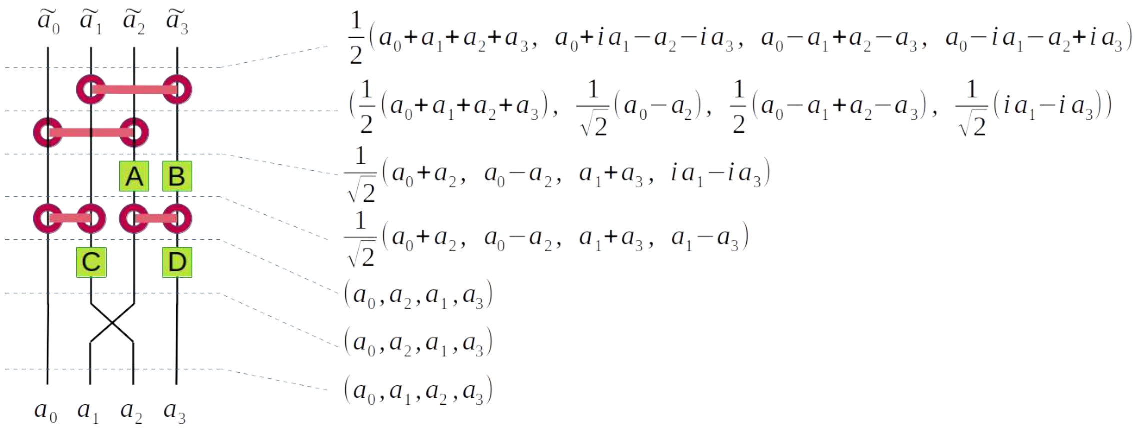


The phase shifts are:

$$A, C, D : e^0 = 1\,, \qquad B : e^{i\pi/2} = i$$

So three of them are essentially identities, only B shifts the phase by a factor of i.

(c) Here is a graphical representation of the initial, the final, and the intermediate values at the vertical lines in the network



As one can see, the total effect of the network coincides with the 4×4 matrix $\mathcal{F}^{(4)}_{ij}$:

$$\begin{pmatrix} \tilde{a}_0 \\ \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \frac{1}{2} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}}_{=\mathcal{F}^{(4)}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Exercise 6.2: Discrete Laplacian (7P)

Let $f : \mathbb{R} \to \mathbb{R}$ be a nice (smooth and differentiable) function. Let us define a linear operator Δ_a, which maps this function onto a different function $g = \Delta_a f$ by means of

$$g(x) \;=\; [\Delta_a f](x) \;=\; \frac{1}{a^2}\Big(f(x+a) - 2f(x) + f(x-a)\Big).$$

This operator, also knwon as discrete Laplacian, is *nonlocal* since it combines values of f at three different positions $(x, x \pm a)$.

(a) Consider the Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \, e^{-ikx} f(x)$$

and likewise $\tilde{g}(k)$. Prove that $\tilde{g}(k)$ and $\tilde{f}(k)$ are related by a *local* map in the sense that $\tilde{f}(k)$ is multiplied by a k-dependent local function. (2P)

(b) Prove that $\Delta_a = \frac{2}{a^2}\left[\cosh\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right) - 1\right]$ and compute the limit $\lim_{a\to 0} \Delta_a f(x)$. (2P)

(c) Compute the limit $\lim_{a\to 0} \frac{4}{a^2}(\Delta_{2a} - \Delta_a)$? (1P)

(d) Finally consider a generalization of Δ_a, namely Δ_a^* defined by

$$g(x) \;=\; [\Delta_a^* f](x) \;=\; \frac{1}{a^2}\sum_{n=-\infty}^{+\infty} b_n\, f(x+an)$$

with (k-independent) coefficients $b_n \in \mathbb{R}$. Determine these coefficients such that $\tilde{g}(k) = -k^2 \tilde{f}(k)$. (2P)

Motivation: Such an operator has the advantage of behaving on the lattice exactly in the same way as the ordinary Laplacian in continuum. However, such an operator is completely non-local. The study of such *perfect lattice operators* was very fashionable some 20 years ago.

Sample Solution

(a) In Fourier space we get the relation: (1P)

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ikx}\tilde{g}(k) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{d}k\, \Delta_a e^{ikx}\tilde{f}(x)$$

Computing $\Delta_a e^{ikx}$ and comparing the integrands we get (1P)

$$\tilde{g}(k) \;=\; \left(\frac{e^{ika}+e^{-ika}-2}{a^2}\right)\tilde{f}(k)$$

The term on the r.h.s. is local: it depends only on k but not on $k\pm 1$.

(b) Using the well-known relation (provable by considering the exp-Taylor series)

$$\exp(a\frac{d}{dx})f(x) = f(x+a)$$

one obtains directly (1P)

$$\Delta_a f(x) = \frac{1}{a^2}\Big(\underbrace{\exp(a\frac{d}{dx}) + \exp(-a\frac{d}{dx})}_{=2\cosh(a\frac{d}{dx})} - 2\Big) f(x)$$

In the limit $a \to 0$ we have $\cosh(a\frac{\mathrm{d}}{\mathrm{d}x}) \simeq 1 - \frac{1}{2}a^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, hence we get the ordinary Laplacian (second derivative) (1P)

$$\lim_{a\to 0}\Delta_a f(x) \;=\; \Delta f(x) \;=\; \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} f(x).$$

(c) The difference between Δ_{2a} and Δ_a in the limit $a \to 0$ can be obtained by Taylor expansion of the result obtained in (b):

$$\Delta_a = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{a^2\,\mathrm{d}^4}{12\,\mathrm{d}x^4} + \mathcal{O}(\frac{a^4\,\mathrm{d}^6}{\mathrm{d}x^6})$$

$$\Rightarrow \qquad \lim_{a\to 0}\frac{4}{a^2}(\Delta_{2a} - \Delta_a) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}$$

(d) To find the coefficients b_n, we first repeat what we have done in part (a):

$$\tilde{g}(k) = \frac{1}{a^2}\Big(\sum_{n=-\infty}^{+\infty} b_n\, e^{iakn}\Big)\tilde{f}(k).$$

What we would like to have is $\tilde{g}(k) = -k^2\tilde{f}(k)$, leading us to the condition (1P)

$$\sum_{n=-\infty}^{+\infty} b_n\, e^{iakn} = -a^2k^2.$$

On the left hand side we have an infinite Fourier series. This can be inverted by

$$b_n = \frac{1}{2\pi a}\int_{-\pi/a}^{+\pi/a} \mathrm{d}k\,(-k^2a^2)\,e^{iakn}\,.$$

Evaluation of this integral (e.g. *Mathematica*®) yields (1P)

$$b_n = -\frac{a}{2\pi}\int_{-\pi/a}^{+\pi/a} \mathrm{d}k\, k^2 e^{iakn} = \begin{cases} -\frac{\pi^2}{3} & \text{if } n = 0 \\ -\frac{2(-1)^n}{n^2} & \text{otherwise} \end{cases}$$

(Σ = 12P)