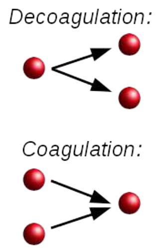
Sample Solutions Exercise 5

Exercise 5.1: Coagulation-decoagulation product state (3P)

Consider the coagulation-decoagulation process with



- hopping to the left $0A \to A0$ at rate w_L,
- hopping to the right $A0 \to 0A$ at rate w_R,
- coagulation $AA \to 0A$ and $AA \to A0$ with rate w_C
- decoagulation $0A \to AA$ and $A0 \to AA$ with rate w_D.

on 1D chain with L sites. All four rates are assumed to be positive.

(a) Set up the two-site Liouville operator $\mathcal{L}^{(2)}$. (1P)

(b) How do we have to tune the rates so that the stationary state is a nontrivial homogeneous product state (nontrivial means that it is neither the fully occupied nor the empty lattice)? How large is the average density of particles in this case? (2P)

Sample Solution

(a) The two-site Liouvillian reads:

$$\mathcal{L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & w_L + w_D & -w_R & -w_C \\ 0 & -w_L & w_D + w_R & -w_C \\ 0 & -w_D & -w_D & 2w_C \end{pmatrix}.$$

(b) We want to tune the rates in such a way that the homogeneous product state (1P)

$$|P_{stat}\rangle = \begin{pmatrix} 1-p \\ p \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1-p \\ p \end{pmatrix} = \begin{pmatrix} 1-p \\ p \end{pmatrix}^{\otimes L}$$

is a stationary state, i.e.

$$\sum_{i=1}^{L-1} \mathcal{L}^{(2)}_{i,i+1} |P_{stat}\rangle = 0.$$

The easiest approach is to look for solutions where each summand in the expression given above vanishes. This leads us to the condition (1P)

$$\mathcal{L}^{(2)} \begin{pmatrix} 1-p \\ p \end{pmatrix}^{\otimes 2} = 0$$

or, equivalently

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & w_L + w_D & -w_R & -w_C \\ 0 & -w_L & w_D + w_R & -w_C \\ 0 & -w_D & -w_D & 2w_C \end{pmatrix} \begin{pmatrix} (1-p)^2 \\ p(1-p) \\ p(1-p) \\ p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This identity comprises three non-trivial equations, two of which are linearly independent (because of the column sum being zero at least one of the equations is automatically linearly dependent on the others). We may e.g. take the second and the last line:

$$(w_L + w_D - w_R)p(1-p) = w_C p^2\,, \quad 2w_D p(1-p) = 2w_C p^2.$$

This system has two solutions: The trivial solution is $p = 0$ (this is indeed a stationary state). The non-trivial solution is (1P)

$$w_L = w_R\,, \quad p = \frac{w_D}{w_C + W_D}.$$

Hence we have to tune the hopping rates such that they are symmetric. The particle density is just $p = \frac{w_D}{w_C + W_D}$.

Exercise 5.2: Matrix product states for the ASEP (9P)

Let us consider the ASEP with particle injection and removal at the boundaries, choosing the hopping rates $w_R = \lambda$ und $w_L = \lambda^{-1}$. In the lecture we have shown that there is a special line parametrized by $\alpha + \beta = \lambda - \lambda^{-1}$, where the matrix product state reduces to an ordinary product state without any correlations. Moreover, it was shown that in general the matrix representations for the ASEP are infinite-dimensional. In this exercise we show that there is another line in the parameter space along which a two-dimensional representation exists. For solving this exercise use *Mathematica®* or similar software.

(a) Prove that for $(\alpha + \lambda^{-1})(\beta + \lambda^{-1}) = 1$ the two-dimensional matrices and vectors

$$\tilde{\mathbf{E}} = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 1 & \frac{1+\alpha\lambda}{\alpha\lambda^2} \end{pmatrix}, \quad \tilde{\mathbf{D}} = \begin{pmatrix} \frac{1}{\beta} & -1 \\ 0 & \frac{1+\beta\lambda}{\beta\lambda^2} \end{pmatrix}, \quad \langle\tilde{\alpha}| = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad |\tilde{\beta}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are a representation of the ASEP algebra (3P)

$$\lambda\tilde{\mathbf{D}}\tilde{\mathbf{E}} - \lambda^{-1}\tilde{\mathbf{E}}\tilde{\mathbf{D}} = \tilde{\mathbf{D}} + \tilde{\mathbf{E}}\,, \quad \langle\tilde{\alpha}|\tilde{\mathbf{E}} = \alpha^{-1}\langle\tilde{\alpha}|\,, \quad \tilde{\mathbf{D}}|\tilde{\beta}\rangle = \beta^{-1}|\tilde{\beta}\rangle\,.$$

(b) Choose λ according to the condition in (a) and compute the four-component vector

$$|P\rangle = \frac{1}{\mathcal{N}}\Big(\langle\tilde{\alpha}|\tilde{\mathbf{E}}\tilde{\mathbf{E}}|\tilde{\beta}\rangle\,, \quad \langle\tilde{\alpha}|\tilde{\mathbf{E}}\tilde{\mathbf{D}}|\tilde{\beta}\rangle\,, \quad \langle\tilde{\alpha}|\tilde{\mathbf{D}}\tilde{\mathbf{E}}|\tilde{\beta}\rangle\,, \quad \langle\tilde{\alpha}|\tilde{\mathbf{D}}\tilde{\mathbf{D}}|\tilde{\beta}\rangle\Big)$$

of a chain with two sites, where

$$\mathcal{N} = \langle\tilde{\alpha}|\tilde{\mathbf{C}}\tilde{\mathbf{C}}|\tilde{\beta}\rangle\,, \quad \tilde{\mathbf{C}} = \tilde{\mathbf{D}} + \tilde{\mathbf{E}}\,,$$

is the normalization. Apply the full 4×4 Liouvillian on a chain with two sites directly to $|P\rangle$ in order to verify that this vector is indeed the stationary state. (3P)

(c) Compute the bare correlation function C_{12} and its connected part C_{12}^{conn} in the stationary state on a chain with two sites. (3P)

Sample Solution

This exercise should be solved with the help of an algebraic computer system:

(a) First we define the relevant equations (here in *Mathematica®* , rename E, D to e, d to avoid name conflict with derivative and exponential) (1P)

```
algebra := {lambda*d.e - 1/lambda*e.d == d + e,
            left.e == left/alpha,
            d.right == right/beta} // Flatten
```

Then define the given representation

```
e = {{1/alpha,  0}, {1, (1 + alpha lambda)/(alpha lambda^2)}};
d = {{1/beta, -1}, {0, (1 + beta lambda)/(beta lambda^2)}};
left = right = {1, 0};
```

Use the command (1P)

```
betasolution = Solve[algebra, beta] // Simplify
lambdasolution = Solve[algebra, lambda] // Simplify
```

to solve the algebra. The solution reads

$$\left\{\beta \to -\frac{-\lambda^2 + \alpha\lambda + 1}{\alpha\lambda^2 + \lambda}\right\}$$

$$\left\{\lambda \to \frac{-\sqrt{\alpha^2 - 2\alpha\beta + \beta^2 + 4} + \alpha + \beta}{2 - 2\alpha\beta}\right\}, \left\{\lambda \to \frac{\sqrt{\alpha^2 - 2\alpha\beta + \beta^2 + 4} + \alpha + \beta}{2 - 2\alpha\beta}\right\}$$

By typing

```
(lambda^-1 + alpha) (lambda^-1 + beta) == 1 /. lambdasolution // Simplify
```

you can convince yourself that these solutions are equivalent to (1P)

$$\left(\alpha + \frac{1}{\lambda}\right)\left(\beta + \frac{1}{\lambda}\right) = 1.$$

(b) The computation of $|P\rangle$ is straight forward. First we compute the norm: (1P)

$$\mathcal{N} = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2 - 1$$

The unnormalized vector reads (1P)

$$\left\{\frac{1}{\alpha^2}, \frac{1}{\alpha\beta}, \frac{1}{\alpha\beta} - 1, \frac{1}{\beta^2}\right\}.$$

Dividing by the norm we obtain the result:

$$|P\rangle = \begin{pmatrix} \frac{1}{\alpha^2\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2-1\right)} \\ \frac{\alpha\beta}{\alpha^2(-\beta^2)+\alpha^2+2\alpha\beta+\beta^2} \\ \frac{\frac{1}{\alpha\beta}-1}{\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2-1} \\ \frac{1}{\beta^2\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2-1\right)} \end{pmatrix}$$

Interestingly, this result does not make use of the condition $(\alpha + \lambda^{-1})(\alpha + \lambda^{-2}) = 1$ from (a).
The Liouvillian of the ASEP reads (1P)

$$\mathcal{L} = \begin{pmatrix} \alpha & -\beta & 0 & 0 \\ 0 & \alpha+\beta+\frac{1}{\lambda} & -\lambda & 0 \\ -\alpha & -\frac{1}{\lambda} & \lambda & -\beta \\ 0 & -\alpha & 0 & \beta \end{pmatrix}$$

It is straight-forward to check that $\mathcal{L}|P\rangle = 0$.

(c) With $\chi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ one can directly compute

$$C_{12} = \langle\Sigma|\chi \otimes \chi|P_{stat}\rangle = \frac{1}{\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2 - 1\right)\beta^2}$$

and the connected correlation function reads:

$$C_{12}^{conn} = C_{12} - \langle\Sigma|\chi \otimes \mathbb{1}|P_{stat}\rangle\langle\Sigma|\mathbb{1} \otimes \chi|P_{stat}\rangle = \frac{\alpha^3\beta^3}{(\alpha+\beta-\alpha\beta)^2(\alpha+\beta+\alpha\beta)^2}.$$

We can also compute the correlators directly in the matrix product formalism:

$$C_{12} = \frac{\langle\alpha|\mathbf{DD}|\beta\rangle}{\langle\alpha|\mathbf{CC}|\beta\rangle} = \frac{1}{\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2 - 1\right)\beta^2}$$

and

$$C_{12}^{conn} = C_{12} - \frac{\langle\alpha|\mathbf{DC}|\beta\rangle\langle\alpha|\mathbf{CD}|\beta\rangle}{\left(\langle\alpha|\mathbf{CC}|\beta\rangle\right)^2} = \frac{\alpha^3\beta^3}{(\alpha+\beta-\alpha\beta)^2(\alpha+\beta+\alpha\beta)^2}.$$

Poitns: (1P) for the correct definition of the correlations, (1P) each for the correct computation of the bare and the connected part.

(Σ = 12P)