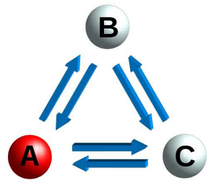
Sample Solutions Exercise 4

Exercise 4.1: Master equation (6P)

Consider a Markov jump process with three possible configurations A, B, and C. Suppose that the rates for this system are given by



$$
\begin{aligned}
w_{A\to B} &= 3\,s^{-1}, \qquad w_{B\to A} = 1\,s^{-1},\\
w_{A\to C} &= 2\,s^{-1}, \qquad w_{C\to A} = 2\,s^{-1},\\
w_{B\to C} &= 1\,s^{-1}, \qquad w_{C\to B} = 3\,s^{-1}.
\end{aligned}
$$

(a) Compute the matrix of the Liouville operator $\mathcal{L}$ in the configuration basis. (1P)

(b) Determine its eigenvalues as well as the left and right eigenvectors. (1P)

(c) Find the stationary probability distribution

$$
|P_{stat}\rangle = \begin{pmatrix} P_A \\ P_B \\ P_C \end{pmatrix}
$$

int the limit $t \to \infty$ and show that in this case the stationary probability currents $J^{stat}_{c\to c'} := P^{stat}_c w_{c\to c'}$ cancel pairwise, i.e., $J_{c\to c'} = J_{c'\to c}$. (1P)

(d) Let the system start in configuration A. Expand the initial probability distribution

$$
|P_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

as a linear combination of the right eigenvectors. (2P)

(e) With this initial configuration compute $P_A(t)$, $P_B(t)$, and $P_C(t)$ explicitly as a function of time in terms of the eigenmode decomposition. (1P)

Sample Solution

(a) Suppressing the units of time, the Liouvillian in the ABC basis is given by

$$
\mathcal{L} = \begin{pmatrix} 5 & -1 & -2 \\ -3 & 2 & -3 \\ -2 & -1 & 5 \end{pmatrix}.
$$

(b) The eigenvalues and left/right eigenvectors read (T denotes the transpose, a column vector):

eigenvalue	left eigenvector	right eigenvector
0	$(1,1,1)$	$(1,3,1)^T$
5	$(3,-2,3)$	$(1,-2,1)^T$
7	$(-1,0,1)$	$(-1,0,1)^T$

(c) To find the stationary probability distribution the right eigenvector to the eigenvalue 0 has to be normalized such that $\langle 1|P_{stat}\rangle = 1$:

$$|P^{stat}\rangle = \begin{pmatrix} 1/5 \\ 3/5 \\ 1/5 \end{pmatrix}.$$

It is now easy to verify that the probability currents cancel one another:

$$J^{stat}_{A\to B} = P^{stat}_A w_{A\to B} = \frac{1}{5}3 = \frac{3}{5}1 = P^{stat}_B w_{B\to A} = J^{stat}_{B\to A}$$
$$J^{stat}_{A\to C} = P^{stat}_A w_{A\to C} = \frac{1}{5}2 = \frac{1}{5}2 = P^{stat}_C w_{C\to A} = J^{stat}_{C\to A}$$
$$J^{stat}_{B\to C} = P^{stat}_B w_{B\to C} = \frac{3}{5}1 = \frac{1}{5}3 = P^{stat}_C w_{C\to B} = J^{stat}_{C\to B}$$

Remark: This is known as the condition of *detailed balance.*

(d) Solving the system of equations

$$a\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

we obtain $a = \frac{1}{5}$ (the normalization already computed above), $b = \frac{3}{10}$, and $c = -\frac{1}{2}$. (Note that different answers are possible since eigenvectors can be rescaled freely and even the total sign can be different.)

(e) Suppressing the unit of time we simply have to insert the corresponding exponential functions:

$$|P(t)\rangle = \begin{pmatrix} P_A(t) \\ P_B(t) \\ P_C(t) \end{pmatrix} = a\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-5t} + c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-7t} = \begin{pmatrix} \frac{1}{5} + \frac{e^{-7t}}{2} + \frac{3e^{-5t}}{10} \\ \frac{3}{5} - \frac{3e^{-5t}}{5} \\ \frac{1}{5} - \frac{e^{-7t}}{2} + \frac{3e^{-5t}}{10} \end{pmatrix}$$

Exercise 4.2: Tensor product ⊗ **(6P)**

Consider two diagonalizable matrices A and B with dimensions $M \times M$ and $N \times N$, respectively. Show that

(a) $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\mathrm{Tr}[B]$. (1P)

(b) $\det(A \otimes B) = \det(A)^N \det(B)^M$. (2P)

(c) $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A)\mathrm{rank}(B)$. (1P)

(d) Let $\sigma^{x,y,z}$ be the usual Pauli matrices. Compute the 4×4 matrix

$$H = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z$$

and compute the eigenvalues of H. Try to explain the degeneracies. (2P)

Sample Solution

(a) This is very simple:

$$\mathrm{Tr}[A \otimes B] = \sum_{k=1}^{MN} (A \otimes B)_k = \sum_{i=1}^{M} \sum_{j=1}^{N} A_i B_j = (\sum_{i=1}^{M} A_i)(\sum_{j=1}^{N} B_j) = \mathrm{Tr}[A]\mathrm{Tr}[B]$$

(b) For the determinant there are several solutions. One is that we diagonalize A and B by transformations U and V such that $A' = UAU^{-1}$ and $B' = VBV^{-1}$ is diagonal, hence

$$A' \otimes B' = (UAU^{-1}) \otimes (VBV^{-1}) = (U \otimes V)(A \otimes B)(U \otimes V)^{-1}$$

is also diagonal. This transformation does not change the determinant, meaning that $\det(A \otimes B) = \det(A' \otimes B')$. Let a_i and b_i be the diagonal elements (eigenvalues) of A' and B'. Then

$$\begin{aligned} \det(A \otimes B) &= \det(A' \otimes B') = \prod_{i=1}^{M} \prod_{j=1}^{N} (a_i b_j) = (\prod_{i=1}^{M} a_i^N)(\prod_{j=1}^{N} b_j^M) \\ &= \det(A')^N \det(B')^M = \det(A)^N \det(B)^M. \end{aligned}$$

Notice in particular the third equality: Powers and sums behave differently. If you pull out a common factor from a sum, you simply place this factor in front of the sum. If you pull out a common factor from a product, you get this factor raised to the power of the number of factors over which the product is carried out.

(c) Clearly, the image of $A \otimes B$ is given by the tensor product of the images:

$$\mathrm{img}(A \otimes B) = \mathrm{img}(A) \otimes \mathrm{img}(B).$$

Since $\mathrm{rank}(A) = \dim(\mathrm{img}(A))$ we have

$$\begin{aligned} \mathrm{rank}(A \otimes B) &= \dim[\mathrm{img}(A \otimes B)] = \dim[\mathrm{img}(A) \otimes \mathrm{img}(B)] \\ &= \dim[\mathrm{img}(A)] \cdot \dim[\mathrm{img}(B)] = \mathrm{rank}(A)\mathrm{rank}(B) \end{aligned}$$

(d)

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues are $(-3, 1, 1, 1)$. Remark: This is a singlet and a triplet, reflecting the underlying SU(2)-symmetry of H.

(Σ = 12P)