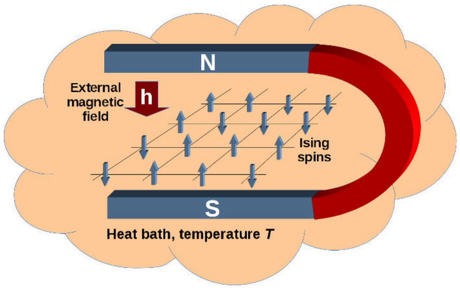
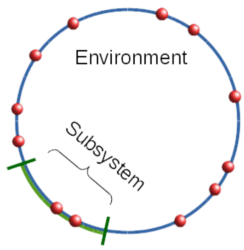


Ising model

Exercise 9.1: Thermalization (7P)



In this exercise we consider again the symmetric exclusion process in 1D. The chain as a whole is considered as a closed thermodynamic system isolated from the environment, i.e., the number of particles is conserved. The aim of this exercise is to show that a small section of this chain, regarded as a subsystem, thermalizes in a grand-canonical ensemble since the complement of the section acts like a reservoir.

Let us consider a 1-dimensional chain with L sites, periodic boundary conditions, and symmetric rates $w_L = w_R = 1$. Assume that the system is populated with $N < L$ particles. Consider a section of K consecutive sites. The aim is to approximate the stationary probability distribution P_s^{stat} in the subsystem, where s denotes the microscopic configuration of the particles in the subsystem.

(a) Find an exact expression for P_s^{stat} in the stationary state (correctly normalized). *Hint:* It will depend on M_s, the number of particles in the subsection. (3P)

(b) Use Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ in order to derive a simpler approximation for probability distribution P_s^{stat}. (2P)

(c) Take this approximation to compute the thermodynamic limit of an infinitely long chain ($L \to \infty$), keeping the particle density $\rho = N/L$ as well as the size of the section K constant. (1P)

(d) How do we have to choose the density ρ in order to get a grand-canonical distribution $P_s^{\text{stat}} \propto e^{-\mu M_s}$, where M_s denotes the number of particles of the subsystem in the configuration s? How does the chemical potential μ depend on ρ? (1P)

Exercise 9.2: Ising meanfield theory (5P)

In the mean field approximation of the Ising model, the average magnetization of a spin $\langle s \rangle$ is given by the implicit equation (see lecture notes)

$$\langle s \rangle \;=\; \tanh(\beta \bar{H}) \;=\; \tanh\bigl[2dJ\beta\langle s \rangle + \beta h\bigr]. \tag{1}$$

The aim of this exercise is to compute the partition sum as well as the critical exponents within the mean field approximation.

Please turn over ⇒

(a) Compute the partition sum $Z(\beta, h)$, keeping $\bar{H}$ (a function that implicitly depends on β and h) in your calculation as it is. (1P)

(b) Show that the partition sum can be written in the form

$$Z(\beta, h) = \left(\frac{2}{\sqrt{1 - \langle s \rangle^2}} \right)^N$$

and give an expression for the thermodynamic potential $\mathcal{V} = \ln Z$. (1P)

(c) Show that the heat capacity can be expressed as (1P)

$$C = N\beta^2 \bar{H}^2 \big(1 - \langle s \rangle^2\big).$$

(d) Expand and solve the implicit equation (1) in a suitable manner that allows you to determine the exponents $\tilde{\beta}$ as well as δ (please avoid confusing the exponent $\tilde{\beta}$ with the inverse temperature β). (2P)

(Σ = 12P)

To be submitted electronically on Wednesday, June 29, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.