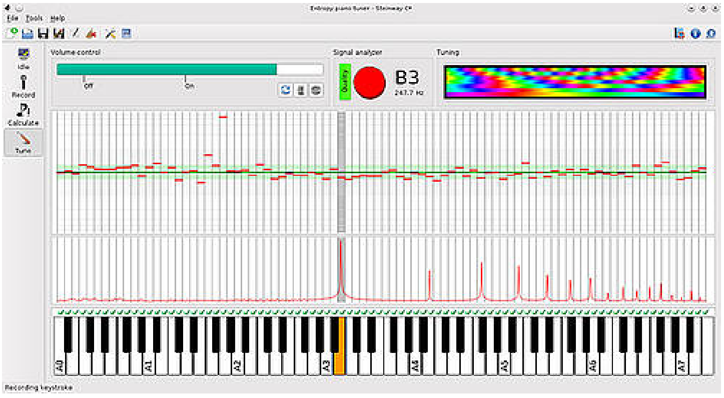
Physics of Complex Systems

Lecture and Tutorials – Prof. Dr. Haye Hinrichsen – B. Sc. Thomas Siedler – SS 2022



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Exercise 8.1: Maximal entropy (3P)

The Shannon entropy of a probability distribution is given by

$$H = -\sum_c p_c \ln p_c \,.$$

(a) Use Jensens inequality for convex functions to show that the Shannon entropy attains its global maximum for a uniform distribution. (1P)

(b) Prove the same statement in the framework of variational calculus. Use the method of Langrange multipliers to take the normalization constraint of the probability distribution into account. (2P)

Exercise 8.2: Master equation of embedded subsystems (3P)

A laboratory system with the configuration space Ω^{sys} is embedded into the environment by means of a projection $\pi : \Omega^{\text{tot}} \mapsto \Omega^{\text{sys}} : c \mapsto s$ (see lecture notes). The total system is ergodic and and has a unique stationary state.

(a) Suppose that the probability distribution $P_c(t)$ of the total system evolves ergodically according to the master equation $\frac{\mathrm{d}}{\mathrm{d}t}P_c(t) = \sum_{c'} P_{c'}(t) w_{c'\to c} - P_c(t)\sum_{c'} w_{c\to c'}$ with symmetric time-independent rates $w_{c'\to c} = w_{c\to c'} \geq 0$. Show that the probability distribution of the laboratory system is exactly given by a master equation with certain effective rates that may be non-symmetric and time-dependent. (1P)

(b) Prove that for a laboratory system in a genuine non-equilibrium steady state (stationary and violating detailed balance), the corresponding total system must be non-stationary and infinitely large. (2P)

Please turn over ⇒

EXERCISE 8.3: MUTUAL INFORMATION **(6P)**

With this exercise we want to recall some basic notions of statistics in bipartite systems, namely, marginal and conditional probabilities, entropies, as well as mutual information (see Wikipedia or any textbook): Consider two random variables $X, Y \in \{0, 1\}$ with the joint probability distribution $P_{XY}(x, y)$ defined by

$$P_{XY}(0,0) = 1/2, \quad P_{XY}(0,1) = 1/4, \quad P_{XY}(1,0) = 0, \quad P_{XY}(1,1) = 1/4\,.$$

Let $P_X(x)$ and $P_Y(y)$ be the corresponding marginal probabilities. Please compute

(a) the joint (total,full) entropy H_{XY}. (1P)

(b) the marginal entropies H_X and H_Y. (1P)

(c) the conditional entropies $H_{X|Y}$ and $H_{Y|X}$. (2P)

(d) the mutual information $I_{X:Y} = H_X + H_Y - H_{XY}$. Verify that the mutual information is also given by $I_{X:Y} = H_X - H_{X|Y} = H_Y - H_{Y|X}$. (2P)

(Σ = 12P)

To be submitted electronically on Wednesday, June 22, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.