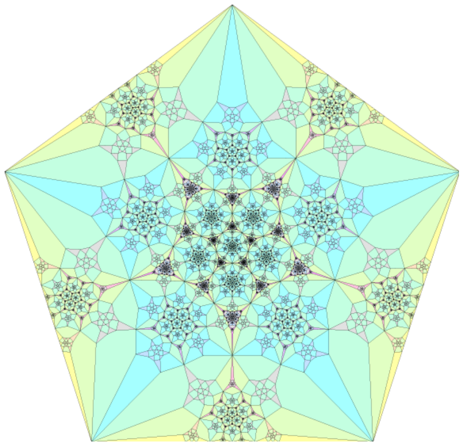
Physics of Complex Systems

Lecture and Tutorials – Prof. Dr. Haye Hinrichsen – B. Sc. Thomas Siedler – SS 2022



Symmetry

Exercise 7.1: Symmetry and degeneracies (5P)

(a) Suppose that the operators $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy the commutation relation $[\mathbf{a}, \mathbf{b}] = \mathbf{c}$. Show that the operators $\mathbf{A} = \mathbf{a} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{a}$, $\mathbf{B} = \mathbf{b} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{b}$, and $\mathbf{C} = \mathbf{c} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{c}$ obey exactly the same commutation relation $[\mathbf{A}, \mathbf{B}] = \mathbf{C}$. (1P)

(b) Assume that $\sigma^+, \sigma^-, \mathbf{m}$ obey the $su(2)$ algebra $[\sigma^+, \sigma^-] = 2\mathbf{m}$ and $[\mathbf{m}, \sigma^\pm] = \pm\sigma^\pm$. Then the operators $\mathbf{S}^\pm = \sigma^\pm \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^\pm$ and $\mathbf{M} = \mathbf{m} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{m}$ obey the same algebra. Show without using a representation that the so-called *Casimir operator*

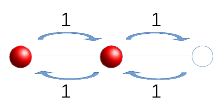
$$\mathcal{C} = \frac{1}{2}(\mathbf{S}^+\mathbf{S}^- + \mathbf{S}^-\mathbf{S}^+) + \mathbf{M}^2$$

commutes with $\mathbf{S}^\pm$ and $\mathbf{M}$. (2P)

(c) Use the Pauli representation $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ and $m = \frac{1}{2}\sigma^z$ in order to verify that the 2-site exclusion process with symmetric rates $w_L = w_R = 1$ and $\alpha = \beta = 0$ is (up to a minus sign and an offset proportional to the identity matrix) equal to the Casimir operator $\mathcal{C}$, proving that this model is $SU(2)$-invariant. (2P)

Exercise 7.2: Spectrum-generating algebra (7P)

Let us consider a three-site symmetric exclusion process ($w_L = w_R = 1$) with closed boundaries as shown in the figure. Its Liouvillian is given by $\mathcal{L} = \mathcal{L}_{12} + \mathcal{L}_{23}$ where



$$X := \mathcal{L}_{12} = \mathcal{L}^{(2)} \otimes \mathbb{1}\,, \quad Y := \mathcal{L}_{23} = \mathbb{1} \otimes \mathcal{L}^{(2)}\,.$$

$\mathcal{L}$ has the eigenvalues $\{0, 0, 0, 0, 1, 1, 3, 3\}$ (on quartet and two doublets). Our aim is to compute the eigenvalues of $\mathcal{L}$ solely by algebraic methods without using any representation, in a similar way as one solves the harmonic oscillator in terms of $a, a^\dagger$.

Please turn over ⇒

(a) Verify that the matrices X and Y obey the so-called *Temperley-Lieb algebra* (1P)

$$\begin{array}{rll} \textit{Idempotence} & X^2 = 2X, & Y^2 = 2Y \\ \textit{Braid group property} & XYX = X, & YXY = Y \end{array}$$

Please use from now on exclusively the symbols X and Y and the four algebraic relations given above. Do not use their explicit matrix representation.

(b) Specify the *monomials* of the algebra, i.e., the elementary words formed by the letters X and Y that cannot be reduced by means of the algebraic relations. (1P)

(c) Joining two monomials by concatenation and applying the algebraic relations one obtains another monomial. List all possible results in a table. (2P)

(d) Now consider a *polynomial*, i.e., a linear combination of all monomials with arbitrary coefficients. Apply the Liouvillian to this polynomial and find the resulting polynomial, showing how the Liouvillian acts in the linear space of polynomials. (1P)

(e) Determine the *eigenpolynomials* of $\mathcal{L} = X + Y$, i.e., those polynomials which are mapped by $\mathcal{L}$ onto themselves up to a factor. Calculate the corresponding eigenvalues and compare them with the spectrum given above. (2P)

(Σ = 12P)

To be submitted electronically on Wednesday, June 15, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.