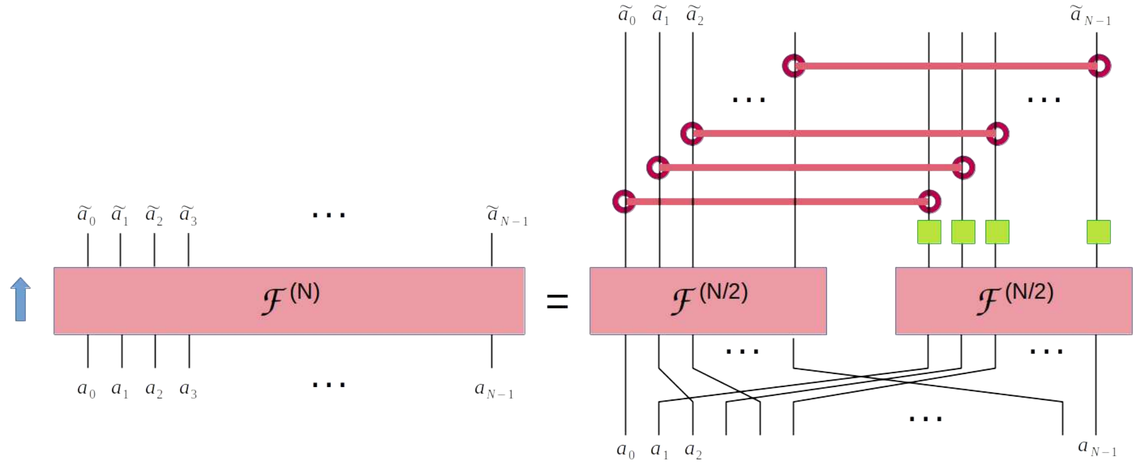
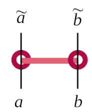
Physics of Complex Systems

Lecture and Tutorials – Prof. Dr. Haye Hinrichsen – B. Sc. Thomas Siedler – SS 2022



Recursion relation of the FFT, represented as a tensor network.

Exercise 6.1: Fast Fourier Transform as a tensor network (5P)



According to the lecture notes the fast Fourier transformation relies on a recursive bisection which can be represented graphically as in the figure shown above.

(a) Explain how the entangler (the red horizontal connection between two line) works, that is, determine $\tilde{a}$ and $\tilde{b}$ as functions of a and a. Make yourself familiar with elementary quantum gates in quantum computing. Which elementary quantum gate does the horizontal line correspond to? (2P)

(b) Let us consider the Fourier transformation $\mathcal{F}^{(4)}$ which acts on four complex numbers $\{a_0, a_1, a_2, a_3\}$. Apply the recursion graphically two times and draw the resulting tensor network (which contains only vertical lines, green phase shifts boxes, and horizontal connections but no red $\mathcal{F}$-boxes). Specify the individual phase shifts of the green boxes. (2P)

(c) At each solid line in the expanded circuit, specify the specific value running along the line in terms of a_0, a_1, a_2, a_3. In particular, express $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ in terms of a_0, a_1, a_2, a_3. Verify that the network indeed realizes a DFT for $N = 4$. (1P)

Exercise 6.2: Discrete Laplacian (7P)

Let $f : \mathbb{R} \to \mathbb{R}$ be a nice (smooth and differentiable) function. Let us define a linear operator Δ_a, which maps this function onto a different function $g = \Delta_a f$ by means of

$$g(x) = [\Delta_a f](x) = \frac{1}{a^2}\Big(f(x+a) - 2f(x) + f(x-a)\Big).$$

This operator, also knwon as discrete Laplacian, is *nonlocal* since it combines values of f at three different positions $(x, x \pm a)$.

Please turn over ⇒

(a) Consider the Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \, e^{-ikx} f(x)$$

and likewise $\tilde{g}(k)$. Prove that $\tilde{g}(k)$ and $\tilde{f}(k)$ are related by a *local* map in the sense that $\tilde{f}(k)$ is multiplied by a k-dependent local function. (2P)

(b) Prove that $\Delta_a = \frac{2}{a^2}\left[\cosh\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right) - 1\right]$ and compute the limit $\lim_{a\to 0} \Delta_a f(x)$. (2P)

(c) Compute the limit $\lim_{a\to 0} \frac{4}{a^2}(\Delta_{2a} - \Delta_a)$? (1P)

(d) Finally consider a generalization of Δ_a, namely Δ_a^* defined by

$$g(x) \;=\; [\Delta_a^* f](x) \;=\; \frac{1}{a^2} \sum_{n=-\infty}^{+\infty} b_n \, f(x + an)$$

with coefficients $b_n \in \mathbb{R}$. Determine these coefficients such that $\tilde{g}(k) = -k^2 \tilde{f}(k)$. (2P)

Motivation: Such an operator has the advantage of behaving on the lattice exactly in the same way as the ordinary Laplacian in continuum. However, such an operator is completely non-local. The study of such *perfect lattice operators* was very fashionable some 20 years ago.

(Σ = 12P)

To be submitted electronically on Wednesday, June 08, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.