Exercise 10.1: The first gap (6P)

The Hamiltonian of the Ising quantum chain is defined by

$$\mathbf{H}_N = \sum_{n=0}^{N-1} \Big(\sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^z \Big)$$

where $\lambda > 0$ and $\sigma_n^{x|y|z} = \mathbb{1}_{2^n \times 2^n} \otimes \sigma^{x|y|z} \otimes \mathbb{1}_{2^{N-n-1} \times 2^{N-n-1}}$ are Pauli matrices acting at site n, assuming periodic boundary conditions. The first (or lowest) gap Δ_N is defined as the difference between the first two lowest-lying eigenvalues of $\mathbf{H}_N$.

(a) In Markov processes, the first gap of $\mathcal{L}$ is the inverse leading relaxation time. In particle physics, the first gap measures the mass. What is the interpretation of the first gap of $\mathbf{H}_N$ in the present context? (1P)

(b) With the *Mathematica®* code fragment for the tensor product $\otimes$ given in the lecture notes,[1] write a code snippet to set up the Hamiltonian for arbitrary N and λ. (2P)
Cross check: The eigenvalues for $N = 3$ (i.e. 8×8) are $\big\{-\lambda - 1, -\lambda - 1, \lambda - 1, \lambda - 1, -2\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1, 2\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1, -2\sqrt{\lambda^2 + \lambda + 1} - \lambda + 1, 2\sqrt{\lambda^2 + \lambda + 1} - \lambda + 1\big\}$

(c) With (b) compute *numerically*[2] the first gap of $\mathbf{H}_N$ for $N = 2, 4, 6, 8, 10, 12$ and and plot the gap for $\lambda = 0.8, 1, 1.2$ double-logarithmically as a function of N. For which λ do you get an almost straight line of points and why?[3] (2P)

(d) Compute the negative *discrete logarithmic derivative*

$$\nu(N) = -\frac{\ln[\Delta_{N+2}/\Delta_N]}{\ln[(N+2)/N]}$$

for $\lambda = 1$ and $N = 2, 4, 6, 8, 10$ and guess the value of the critical exponent ν which is defined as the limit $\nu = \lim_{N\to\infty} \nu(N)$. (1P)

[1] You are of course free to use any other algebraic computer systems as well.
[2] In Mathmatica: Use `N[...]` to make $\mathbf{H}$ numerical: `Eigenvalues[N[H[length,lambda]]]`
[3] The execution for $N = 12$ should take less than 20 seconds. If not, go only up to $N = 10$.

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Exercise 10.2: Duality of the anisotropic XY chain (6P)

Duality requires AdS/CFT? Not at all! There are much simpler toy models which exhibit a duality between bulk and boundary behavior. One such example is the anisotropic XY quantum chain. The Hamiltonian on a 1D chain of L sites with open boundaries is defined by

$$\mathbf{H}(\eta,q) = -\frac{1}{2}\sum_{j=1}^{L-1}\Big(\eta\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y + q\sigma_j^z + q^{-1}\sigma_{j+1}^z\Big)\,,$$

where $\eta > 0$ controls the bulk anisotropy while $q > 0$ controls the boundary terms (since in the bulk the combination $\frac{1}{2}(q+q^{-1})\sigma^z$ prevails).

(a) Let us write $\mathbf{H}(\eta,q) = \sum_{j=1}^{L-1} e_j$, where e_j is the 2-site Hamiltonian acting on sites j and $j+1$. Show that the e_j obey the algebra (2P)

$$e_j^2 = A\,\mathbb{1}_L\,, \quad \Big(e_je_{j+1}e_j - e_{j+1}e_je_{j+1} + (A-1)(e_j - e_{j+1})\Big)\Big(e_j - e_{j+1}\Big) = B\,\mathbb{1}_L\,,$$

where A, B are constants which depend on η, q while $\mathbb{1}_L$ is the $2^L \times 2^L$ unit matrix.

(b) Analyze how A and B depend on η and q. Why is it justified to conjecture that there should exist a similarity transformation $\mathbf{U}$ which swaps these parameters in such a way that (1P)

$$\mathbf{H}(\eta,q) \;=\; \mathbf{U}\,\mathbf{H}(q,\eta)\mathbf{U}^{-1}\ ?$$

(c) The *Jordan Wigner transformation* is defined as

$$\tau_j^{x,y} \;=\; \Big(\prod_{i=1}^{j-1}\sigma_j^z\Big)\sigma_j^{x,y}.$$

The purpose of this transformation is that these operators anticommute also on different sites. Express the Hamiltonian in terms of τ_j^x and τ_j^y. (2P)

(d) Show e.g. with *Mathematica*® that for a 3-site system the similarity transformation is of the form

$$\mathbf{U} = \mathbb{1} + \omega \sum_{1\le j<k\le 3} \tau_j^x\tau_k^x\,.$$

Determine the unknown constant ω as a function of η and q. (1P)

> **Remark:** Explicit expressions for $\mathbf{U}$ can be found for any system size, establishing an exact duality for all η, q, L.

(Σ = 12P)

To be submitted electronically on Wednesday, July 06, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.