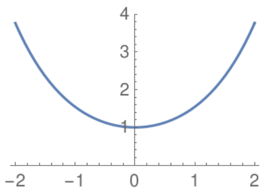
Sample Solutions Exercise 2

Exercise 2.1: Transforming probability densities (6P)

Consider the curve $y = \cosh x$. The aim of this exercise is to decorate this curve in the interval $x \in [x_1, x_2]$ with random points in such a way that the density of the points is uniform along the curve. Since the slope of the curve varies, this means that the x-coordinates of these points are *not* uniformly distributed.



(a) Compute the probability density $p(x)$ of the x-coordinates. Hint: the density of the points per arc length of the curve has to be constant. (2P)

(b) Normalize the probability density on the interval $x \in [x_1, x_2]$. (1P)

(c) Find a function $f : z \mapsto x = f(z)$ such that it maps a uniform probability density $p(z) = \mathit{const}$ to the non-uniform probability density $p(x)$ calculated in (a)-(b). (2P)

(d) Adjust the integration constant in (c) in such a way that f maps $[z_1, z_2] \mapsto [x_1, x_2]$ with $f(z_1) = x_1$ and $f(z_2) = x_2$. Specialize the result for a standard random number $(z_1 = 0, z_2 = 1)$. (1P)

Sample Solution

(a) The infinitesimal line element $\mathrm{d}\ell$ of the curve $y = f(x) = \cosh x$ is given by (1P)

$$\mathrm{d}\ell = \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \sqrt{1 + \sinh^2(x)}\,\mathrm{d}x = \cosh(x)\,\mathrm{d}x$$

We start with the ansatz

$$|p(\ell)\,\mathrm{d}\ell| = |p(x)\,\mathrm{d}x|$$

where $p(\ell) = C$ is constant along the curve. Using this ansatz we get (1P)

$$p(x) = C\left|\frac{\mathrm{d}\ell}{\mathrm{d}x}\right| = C\cosh x.$$

(b) The normalization can be computed straight-forwardly:

$$\int_{x_1}^{x_2} p(x)\,\mathrm{d}x = C(\sinh x_2 - \sinh x_1) \quad \Rightarrow \quad C = \frac{1}{\sinh x_2 - \sinh x_1}$$

giving

$$p(x) = \frac{\cosh x}{\sinh x_2 - \sinh x_1}.$$

(c) Again we start with

$$|p(z)\,\mathrm{d}z| = |p(x)\,\mathrm{d}x|,$$

where $z \in [z_1, z_2]$ is uniformly distributed, meaning that $p(z) = \frac{1}{z_2 - z_1}$ is constant. Again we get a first-order differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z'(x) = \frac{p(x)}{p(z)} = \frac{z_2 - z_1}{\sinh x_2 - \sinh x_1} \cosh x \,,$$

hence

$$z(x) = \frac{z_2 - z_1}{\sinh x_2 - \sinh x_1} \sinh x + \tilde{C},.$$

where $\tilde{C}$ is the integration constant. However, what we need is not $z(x)$ but the inverse function $x(z)$:

$$x(z) = \operatorname{arcsinh}\Big[\frac{\sinh x_2 - \sinh x_1}{z_2 - z_1}\,(z - \tilde{C})\Big]$$

(d) Assume that f maps $[z_1, z_2]$ onto $[x_1, x_2]$ with

$$x(z_1) = z_1 \,, \quad x(z_2) = z_2 \,.$$

It is easy to see that the difference $\sinh(x(z_2)) - \sinh(x(z_1) = \sinh(x_2 - \sinh(x_1)$ is always true, hence both equations are not independent. Taking one of them we can compute the integration constant

$$\tilde{C} = \frac{z_2 \sinh x_1 - z_1 \sinh x_2}{\sinh x_2 - \sinh x_1} \,.$$

Plugging that in we arrive at the final result

$$x(z) = \operatorname{arcsinh}\Big[\frac{(z - z_1)\sinh x_2 - (z - z_2)\sinh x_1}{z_2 - z_1}\Big] \,.$$

For a standard random number generator $z \in [0, 1]$ with $z_1 = 0$ and $z_1 = 1$ this specializes to

$$x(z) = \operatorname{arcsinh}\Big[\sinh x_1 + z\big(\sinh x_2 - \sinh x_1\big)\Big] \,.$$

Exercise 2.2: Normal distribution (6P)

The probability density function of the normal distribution with zero mean and variance σ^2 is given by

$$p(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}} \,.$$

(a) Derive a recursion relation for the moments which expresses m_n in terms of m_{n-2}. The relation can be derived by partial integration of the defining integral. (2P)

(b) Apply this recursion relation to compute the first six moments $m_1, \ldots, m_6$. (1P)

(c) Derive the moment-generating function of the normal distribution given above. Hint: Try quadratic completion (quadratische Ergänzung) in the integrand. (2P)

(d) Compute all cumulants κ_n of the normal distribution given above from the cumulant-generating function. (1P)

Sample Solution

(a) The definition of the n-th moment of a continuous pdf reads

$$\langle x^n \rangle = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x$$

Integrating the right hand side (rhs) by parts we obtain

$$\langle x^n \rangle = \left[\frac{1}{\sigma\sqrt{2\pi}} \frac{1}{n+1} x^{n+1} e^{-\frac{x^2}{2\sigma^2}} \right]_{-\infty}^{+\infty} - \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{n+1} x^{n+1} \frac{-x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x.$$

Since the exponential part vanishes faster than any polynomial, the boundary terms do not contribute. The remaining expression reads

$$\langle x^n \rangle = \frac{1}{\sigma^2(n+1)} \langle x^{n+2} \rangle$$

or equivalently (shifting $n \to n-2$):

$$\langle x^n \rangle = \sigma^2(n-1) \langle x^{n-2} \rangle \qquad \Rightarrow \qquad m_n = \sigma^2(n-1) m_{n-2}\,.$$

(b) By symmetry all odd moments vanish, hence we only have to compute the even moments. The zeroth moment is the norm $m_0 = 1$, which anchors the recursion. Applying the recursion relation step by step we obtain:

$$\begin{aligned} m_2 &= \sigma^2 \\ m_4 &= 3\sigma^4 \\ m_6 &= 5 \cdot 3 \cdot \sigma^6 = 15\sigma^6 \end{aligned}$$

From here we can guess and prove the following solution (not required):

$$m_n = (n-1)(n-3)\cdots 3 \cdot 1 \cdot \sigma^n = \frac{n!}{(n/2)!2^{n/2}} \sigma^n$$

(c) We start with the definition of the moment-generating function

$$M(t) = \langle e^{xt} \rangle = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{xt} e^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x$$

The integral on the rhs can be calculated by completion of the square (quadratische Ergänzung). To this end we shift the integration variable by $x \to x + \mu$, getting

$$M(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{xt+\mu t - \frac{x^2+2\mu x+\mu^2}{2\sigma^2}}\,\mathrm{d}x$$

The method of completing the square amounts to choosing μ in such a way that the two terms which are linear in x cancel. This can be achieved by setting $\mu = t\sigma^2$. In addition, the x-independent parts can be pulled out. The remaining expression reads

$$M(t) = e^{t^2\sigma^2 - \frac{t^2\sigma^4}{2\sigma^2}} = e^{\frac{1}{2}t^2\sigma^2}\,.$$

It is straightforward to verify that the moments computed in part (b) are correctly reproduced by this function (not required).

(d) The cumulant-generating function of the normal distribution is particularly simple:

$$K(t) = \ln M(t) = \frac{1}{2}\sigma^2 t^2$$

Therefore, the normal distribution (with zero mean) possesses only a single non-vanishing cumulant, namely, $\kappa_2 = \sigma^2$, which is just the variance. This is in fact the most salient feature of the normal distribution: It has no properties (such as skewness and curtosis) except for the variance.

(Σ = 12P)