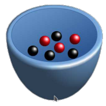
Sample Solutions Exercise 1

Exercise 1.1: Bayes Theorem (2P)

An urn contains three red and four black balls. Two balls are randomly drawn from the urn (without putting them back). What is the probability that the first one is black given that the second one is red?



Sample Solution

Let A be the event that the first drawn ball is black and B that the second drawn ball is red. As we have initially 4 black and 3 red balls in the urn, the probability for the first ball to be black is $P(A) = 4/7$. Likewise the probability that the second ball is red (not knowing the first one) is $P(B) = 3/7$. The conditional probability $P(B|A)$ is the probability that the second one is red, given that the first one was black. This means that there are still three black and three red balls in the urn, hence $P(B|A) = 1/2$. The 'reversed' conditional probability $P(A|B)$ can be computed using Bayes theorem

$$P(A|B) = \frac{P(A)}{P(B)} P(B|A) = \frac{2}{3}.$$

Exercise 1.2: Poisson distribution (6P)

The poisson distribution $P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ can be understood as the limit of the binomial distribution in the case of "rare events".

(a) Let $p = \lambda/N$ and take $N \to \infty$ while keeping λ and k constant. Show that in this limit we can approximate $(1-p)^{N-k} \approx e^{-\lambda}$. (1P)

(b) Show similarly that $\binom{N}{k} \approx \frac{N^k}{k!}$. (1P)

(c) Use (a) and (b) to show that in this limit the binomial distribution tends to the Poisson distribution. (1P)

(d) Check that the Poisson distribution is properly normalized. (1P)

(e) Compute the moment- and cumulant-generating functions. (1P)

(f) Determine all cumulants. (1P)

Sample Solution

(a) Let us consider the logarithm of the left-hand side $(N-k)\ln(1-p)$. Since $p \to 0$ for $N \to \infty$ we can approximate

$$\ln(1-p) \approx -p + \mathcal{O}(p^2).$$

Therefore $(N-k)\ln(1-p) \approx -Np+kp \approx -Np = -\lambda$, hence $(1-p)^{N-k} \approx e^{-\lambda}$

(b) For this we have to show that $N!/(N-k)! \approx N^k$. The take the logarithm and apply Stirlings formula

$$\ln N! - \ln(N-k)! \;\approx\; N\ln N - N - (N-k)\ln(N-k) + (N-k)$$

Since $k \ll N$ we can further approximate $\ln(N-k) \approx \ln N - k/N$. Inserting this approximation in the formula given above and simplifying the expression we arrive at

$$\ln N! - \ln(N-k)! \approx k \ln N.$$

Exponentiating this we finally obtain $\binom{N}{k} \approx \frac{N^k}{k!}$.

(c) Simply insert the results from (a) and (b):

$$\binom{N}{k} p^k (1-p)^{(N-k)} \approx \frac{N^k}{k!} p^k e^{-\lambda} = \frac{\lambda^k e^{-\lambda}}{k!}$$

(d) Check normalization:

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \;=\; e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \;=\; e^{-\lambda} e^{+\lambda} \;=\; 1.$$

(e) The moment-generating function can also be found in the lecture notes:

$$M(t) \;=\; \langle e^{kt} \rangle \;=\; \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{kt} \;=\; e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \;=\; e^{\lambda(e^t-1)}.$$

$$K(t) = \ln M(t) = \lambda(e^t - 1).$$

(f) Take the derivative of the cumulant-generating function:

$$\kappa_n \;=\; \frac{\mathrm{d}^n}{\mathrm{d}t^n} K(t)\Big|_{t=0} \;=\; \lambda e^t \Big|_{t=0} \;=\; \lambda.$$

Therefore, the Poisson distribution is special in so far as all cumulants coincide.

Exercise 1.3: Reconstruction of a probability density (4P)

(a) Prove the following statement: If the moment-generating function $M_X(t)$ is analytic, then the corresponding probability density $p(x)$ is given by the inverse Fourier transform (1P)

$$p(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s \, e^{-ixs} M_X(is).$$

(b) Consider a probability distribution with the cumulants

$$\{\frac{\kappa_0}{0!}, \frac{\kappa_1}{1!}, \frac{\kappa_2}{2!}, \frac{\kappa_3}{3!}, \dots\} \;=\; \{0,\, 0,\, \frac{3}{2},\, 0,\, -\frac{1}{2},\, 0,\, \frac{1}{3},\, 0,\, -\frac{1}{4},\, 0,\, \frac{1}{5},\, 0,\, -\frac{1}{6}, \dots\}$$

Compute the generating functions $K(t)$ and $M(t)$. (2P)

(c) Use (a) to reconstruct the probability density $p(x)$. (1P)

Sample Solution

(a) The MGF is defined as $M(t) = \langle e^{tx} \rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x) e^{tx}$, where $t \in \mathbb{R}$. If $M(t)$ was analytic, this would mean that the defining relation is valid everywhere in the complex plane, in particular on the imaginary line, i.e.

$$M_X(is) = \langle e^{isx} \rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x) e^{isx}\,,$$

where $s \in \mathbb{R}$. This implies that $M(is)$ is (up to a possible prefactor) the Fourier transform of the probability distribution. Thus we can invert the above relation by (1P)

$$p(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} M_X(is).$$

(b) Obviously the nonzero cumulants are even and given by

$$\kappa_{2m} \;=\; \begin{cases} \frac{1}{2} + \frac{(-1)^{m+1}}{m} & \text{if } m = 1 \\ \frac{(-1)^{m+1}}{m} & \text{if } m = 2, 3, \dots \end{cases}.$$

Hence the CGF is given by (1P)

$$K(t) \;=\; \sum_{n=0}^{\infty} \frac{\kappa_n t^n}{n!} \;=\; \tfrac{1}{2} t^2 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} t^{2m}}{m}$$

Since the infinite sum is of the form $\sum_{k=1}^{\infty} \frac{a^k}{k} = -\ln(1-a)$ we end up with

$$K(t) \;=\; \frac{t^2}{2} + \ln(1+t^2)\,,$$

implying that (1P)

$$M(t) \;=\; \exp\big(K(t)\big) \;=\; e^{\frac{1}{2}t^2}(1+t^2).$$

(c) Using the formula from (a) the probability density $p(x)$ reads

$$p(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} M(is) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} e^{-\frac{1}{2}s^2} (1-s^2)$$

We can evaluate this expression with *Mathematica*® , or we can do it by hand as follows: We can express the $-s^2$ contribution in the integrand by a derivative:

$$p(x) \;=\; \left(1 + \frac{\partial^2}{\partial x^2}\right) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} e^{-\frac{1}{2}s^2}.$$

Using standard methods of quadratic completion the integral equals

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-ixs} e^{-\frac{1}{2}s^2} \;=\; \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Taking the derivatives we end up with the final result (1P)

$$p(x) \;=\; \frac{x^2 e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

(Σ = 12P)