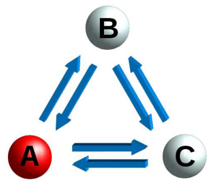
```
In[1]:= Attributes[CircleTimes] = {Flat, OneIdentity};
        CircleTimes[a_List /; VectorQ[a], b_List /; VectorQ[b]] :=
          Flatten[KroneckerProduct[a, b]];
        CircleTimes[a_List /; MatrixQ[a], b_List /; MatrixQ[b]] :=
          KroneckerProduct[a, b];

In[4]:= {a, b} ⊗ {c, d, e}

Out[4]= {a c, a d, a e, b c, b d, b e}

In[5]:= sx = {{0, 1}, {1, 0}};
        sx ⊗ sx

Out[6]= {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}
```

Minimal code for the implementation of a tensor product in *Mathematica*®

Exercise 4.1: Master equation (6P)

Consider a Markov jump process with three possible configurations A, B, and C. Suppose that the rates for this system are given by



$$
\begin{aligned}
w_{A\to B} &= 3\,s^{-1}, \qquad & w_{B\to A} = 1\,s^{-1},\\
w_{A\to C} &= 2\,s^{-1}, \qquad & w_{C\to A} = 2\,s^{-1},\\
w_{B\to C} &= 1\,s^{-1}, \qquad & w_{C\to B} = 3\,s^{-1}.
\end{aligned}
$$

(a) Compute the matrix of the Liouville operator $\mathcal{L}$ in the configuration basis. (1P)

(b) Determine its eigenvalues as well as the left and right eigenvectors. (1P)

(c) Find the stationary probability distribution

$$|P_{stat}\rangle = \begin{pmatrix} P_A \\ P_B \\ P_C \end{pmatrix}$$

int the limit $t \to \infty$ and show that in this case the stationary probability currents $J^{stat}_{c\to c'} := P^{stat}_c w_{c\to c'}$ cancel pairwise, i.e., $J_{c\to c'} = J_{c'\to c}$. (1P)

(d) Let the system start in configuration A. Expand the initial probability distribution

$$|P_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

as a linear combination of the right eigenvectors. (2P)

(e) With this initial configuration compute $P_A(t)$, $P_B(t)$, and $P_C(t)$ explicitly as a function of time in terms of the eigenmode decomposition. (1P)

Please turn over ⇒

EXERCISE 4.2: TENSOR PRODUCT $\otimes$ **(6P)**

Consider two diagonalizable matrices A and B with dimensions $M \times M$ and $N \times N$, respectively. Show that

(a) $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\mathrm{Tr}[B]$. (1P)

(b) $\det(A \otimes B) = \det(A)^N \det(B)^M$. (2P)

(c) $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A)\mathrm{rank}(B)$. (1P)

(d) Let $\sigma^{x,y,z}$ be the usual Pauli matrices. Compute the 4×4 matrix

$$H = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z$$

and compute the eigenvalues of H. Try to explain the degeneracies. (2P)

(Σ = 12P)

To be submitted electronically on Wednesday, May 25, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.