Joseph Liouville 1809-1882

Exercise 3.1: Vector notation and Liouville operator (6P)

Consider a continuous-time Markov process (see lecture notes) with the master equation

$$\dot{P}_c(t) = \sum_{c'} P_{c'}(t) w_{c' \to c} - P_c(t) \sum_{c'} w_{c \to c'} .$$

(a) Prove that the matrix elements of the Liouville operator $\mathcal{L}$ are given by (3P)

$$\langle c' | \mathcal{L} | c \rangle \;=\; -w_{c \to c'} + \delta_{c,c'} \sum_{c''} w_{c \to c''}$$

(b) Consider a system with three configurations $1, 2, 3$ with the rates

$$w_{1\to 2} = 1\,, \quad w_{2\to 1} = \frac{1}{2}\,, \quad w_{2\to 3} = 1\,, \quad w_{3\to 1} = 5\,,$$

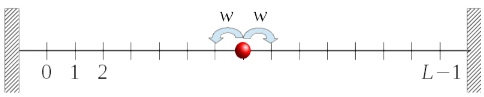
in units of $1/\text{time}$ (all other rates are zero). Determine the matrix of the Liouville operator. (1P)

(c) Determine the so-called zero vectors, i.e., the left and right eigenvectors to the eigenvalue zero. (1P)

(d) Normalize the right zero vector in order to interpret it as a probability distribution. Why are the components of the left and the right eigenvector numerically different? (1P)

Exercise 3.2: One-dimensional random walk (6P)

Consider a particle on a one-dimensional chain with L sites with closed ends performing a symmetric random walk at a constant rate w. Let us enumerate the sites with the index $j = 0, \dots, L-1$.



(a) Find the $L \times L$ matrix of the Liouville operator $\mathcal{L}$ in the configuration basis. (1P)

(b) Consider the eigenvalue problem $\mathcal{L}|\psi_n\rangle = \lambda_n |\psi_n\rangle$ and show that the ansatz

$$(\psi_n)_j = \langle c_j | \psi_n \rangle = A e^{i k_n j} + B e^{-i k_n j}$$

evaluated in the bulk of the chain leads to the dispersion relation $\lambda_n = 4w \sin^2 \frac{k_n}{2}$. (Here $\langle c_j|$ denote the canonical basis vectors). (2P)

Please turn over ⇒

(c) Take the boundary conditions (the first and the last line of the matrix) into account in order to determine all eigenvectors and eigenvalues of $\mathcal{L}$. (2P)

(d) What is the longest typical time scale occurring in the dynamics? How does it depend on L? (1P)

(Σ = 12P)

To be submitted electronically on Wednesday, May 18, 2022, via WueCampus according to our guidelines on the web page `cs.hayehinrichsen.de`.