SAMPLE SOLUTIONS EXERCISE 9

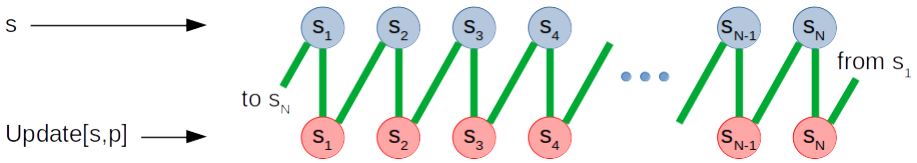
EXERCISE 9.1: NUMERICAL SIMULATION OF DIRECTED PERCOLATION (6P)

The aim of this exercise is to simulate a directed bond percolation process numerically. This can be done in *Mathematica*® (and we will use a Mathematica-like syntax here) but you are free to use any other programming language as well (such as Python or C++). However, you are requested to keep the code compact and understandable so that it can be corrected easily.

(a) Let S be a list of N elements $S = \{s_1, s_2, \ldots, s_N\}$ representing the state of a horizontal row in a directed bond percolation process, where

$$s_i = \begin{cases} 1 & \text{if the site is occupied/active/wet/infected} \\ 0 & \text{if the site is empty/inactive/dry/healthy} \end{cases}$$

Write a function `update[S,p]` that returns the updated row according to the following update scheme:



Here the green lines stand for bonds that are open with probability p and closed otherwise (compared to the lecture notes, the bonds are tilted to the left in order to realize the update on a single list). Note that the leftmost and the rightmost site are connected by a green line, implementing periodic boundary conditions. (2P)

(b) Write another function that returns the density of active sites of the list S. (1P)

(c) Perform a single run of the simulation at the critical point: Start the simulation with a fully occupied lattice $S = \{1, 1, 1, \ldots, 1\}$ with $N = 1000$ sites. Simulate the process for $p = 0.6447$ over $T = 1000$ time steps and plot the density double-logarithmically as a function of time (for example using `ListLogLogPlot`). (1P)

(d) Compute the mean value of the density as a function of time averaged over $M = 100$ stochastically independent runs (this can take several minutes) and plot the average double-logarithmically as a function of time. (1P)

(e) Fit the function $\rho(t) = At^{-B}$ to the average data obtained in (d) estimating the constants A, B. Hint: In *Mathematica*® you can simply use the tool `FindFit`. (1P)

SAMPLE SOLUTION

(a) There are many possible solutions. One of them is the following: (2P)

```
update[S_, p_] := Module[{n, Snew},
n = Length[S]; (* Length of the list *)
Snew = Table[0, {n}]; (* Create new list of same length with zeroes *)
Do[If[s[[i]] == 1, (* Iterate over all active sites in the upper row *)
    (* Vertical bond *)
    If[Random[] < p, Snew[[i]] = 1];
    (* Tilted bond, taking periodic b.c. into account *)
    If[Random[] < p , If[i > 1, Snew[[i - 1]] = 1, Snew[[n]] = 1]];
    ], {i, 1, n}];
Snew]
```

(b) The density in the list is the sum of all entries divided by its length: (1P)

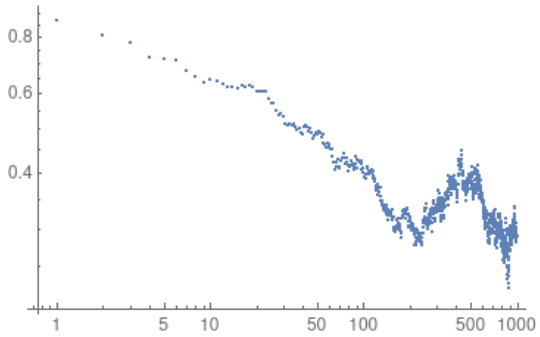
```
density[S_] := N[Total[S]/Length[S]];
```

(The `N[..]` is optional. It converts to numerical expressions.)

(c) Realization in *Mathematica*® : (1P)

```
s = Table[1, {1000}];
T = 1000;
run = Table[s = update[s, 0.6447]; density[s], {T}];
ListLogLogPlot[run]
```

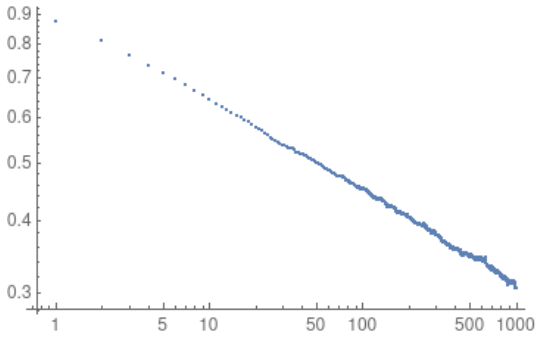
The result should typically look like this:



(d) Realization in *Mathematica*® : (1P)

```
T = 1000;
M = 100;
average = 0;
Do[s = Table[1, {1000}];
   average += Table[s = update[s, 0.6447]; density[s], {T}], {M}
   ]
average /= M;
ListLogLogPlot[average]
```

The result should typically look like this:



(e) In *Mathematica*® we can compute the fit by (1P)

```
FindFit[average, A t^(-B), {A, B}, t]
```

The result reads e.g.:

$$\{A- > 0.921058, B- > 0.156795\}$$

Correction advice: The constant A should be expected to be in the range $[0.85, 1]$. The constant B is the exponent $\delta \approx 0.159$. Here we expect a correct result to be in the range $B \in [0.13, 0.17]$.

Exercise 9.2: DP mean field approximation (6P)

In the mean field approximation of directed percolation (DP) extended by a term for diffusion, the density of active sites $\rho(\mathbf{x}, t)$ evolves according to the partial differential equation

$$\dot{\rho} = \lambda\rho(1-\rho) - \rho + D\nabla^2\rho.$$

(a) Determine the homogeneous (space-independent) stationary (time-independent) density ρ_{stat} as a function of λ. Where is the critical point λ_c? Test the solutions for stability by a stability analysis. (2P)

(b) Find the exponent for $\rho_{stat} \sim (\lambda - \lambda_c)^\beta$ near the critical point. (1P)

(c) At criticality $\lambda = \lambda_c$ the homogeneous density $\rho(t)$ decays as $\rho(t) \sim t^{-\delta}$. Compute the exponent δ. (1P)

(d) Show that the Greens function of the mean field equation at the critical point be written in the form

$$G(\mathbf{x}, t) = t^{-1} f(x^2/t).$$

Hint: It is sufficient to derive an autonomous differential equation for f, you don't have to compute f explicitly. Restrict yourself to $t > 0$ and ignore the scaling of the δ-functions. (2P)

Sample Solution

(a) Stationary means that $\dot{\rho} = 0$. Homogeneous means that $\nabla\rho = 0$. Therefore, we have to solve the quadratic equation $0 = \lambda\rho - \lambda\rho^2 - \rho$ with the solutions (1P)

$$\rho_1 = 0, \quad \rho_2 = 1 - \frac{1}{\lambda}.$$

As one can see, the critical point is $\lambda_c = 1$ since here ρ_2 vanishes. To test the stability let us consider a small perturbation $\rho \to \rho' = \rho + \epsilon$. Then (1P)

$$\dot{\rho}_1' = \lambda(\rho+\epsilon)(1-\rho-\epsilon)\,, \qquad \dot{\rho}_2' = \epsilon(\lambda - (1-1/\lambda)\lambda 2 - 1) + \mathcal{O}(\epsilon^2)\,,$$

giving

$$\dot{\rho}_1' = \epsilon(\lambda - 1) + \mathcal{O}(\epsilon^2)\,, \qquad \dot{\rho}_2' = \epsilon(1-\lambda) + \mathcal{O}(\epsilon^2)\,.$$

Stable are those solutions where the right hand side for positive ϵ is negative. Hence the absorbing state $\rho_1 = 0$ is stable for $\lambda < \lambda_c = 1$ while the active state $\rho_2 = \lambda$ is stable for $\lambda > \lambda_c = 1$. At the transition $\lambda = \lambda_c = 1$ both solutions are marginal. (1P)

(b) The stationary density reads

$$\rho_{\text{stat}} = 1 - \frac{1}{\lambda} = \lambda^{-1}(\lambda - 1) = \frac{1}{\lambda}(\lambda - \lambda_c) \sim (\lambda - \lambda_c)^1 \quad \Rightarrow \quad \beta = 1.$$

(c) For a homogeneous (=position-independent) density the equation at the critical point reduces to

$$\dot{\rho} = -\rho^2 \quad \Rightarrow \quad \rho(t) = \frac{1}{t}$$

This means that $\delta = 1$ in the mean field approximation. (1P)

(d) The propagator should obey the differential equation

$$\dot{G} - \lambda G + G^2 - D\nabla^2 G = \delta^d(\mathbf{x})\delta(t)$$

For solving this exercise it is sufficient to consider this equation for $t > 0$ so that the right hand side is zero. Now we insert the ansatz given in the exercise and obey

$$-\frac{f(x^2/t)}{t^2} - \frac{x^2 f'(x^2/t)}{t^3} + \lambda\frac{f(x^2/t)}{t} + \frac{f^2(x^2/t)}{t^2} - \frac{2Df'(x^2/t)}{t^2} - \frac{4Dx^2 f''(x^2/t)}{t^3} = 0$$

Here we have to set $\lambda = \lambda_c = 0$. Moreover, two terms cancel out. Thus we are left with

$$-\frac{x^2 f'(x^2/t)}{t^3} - \frac{2Df'(x^2/t)}{t^2} - \frac{4Dx^2 f''(x^2/t)}{t^3} = 0$$

Now we multiply the equation with $-t$ and replace $x^2/t = \zeta$. Then we obtain

$$\zeta f'(\zeta) + 2Df'(\zeta) + 4D\zeta f''(\zeta) = 0.$$

This is an autonomous differential equation in f in ζ alone, i.e., x and t do no longer appear in explicitly. This justifies the scaling ansatz given in the exercise.

(Σ = 12P)