SAMPLE SOLUTIONS EXERCISE 10

EXERCISE 10.1: STOCHASTIC PDES: ITO AND STRATONOVIC (5P)

For the Langevin equation $\dot{y}(t) = y(t)\xi(t)$ with the initial condition $y(0) = 1$, we would like to show that the two numerical update schemes

$$\begin{aligned} \text{Ito:} \quad y_{j+1} &= y_j + \sqrt{\Delta t} y_j z_j \\ \text{Stratonovic:} \quad y_{j+1} &= y_j + \sqrt{\Delta t}\frac{1}{2}\Big(y_j + y_{j+1}\Big) z_j \end{aligned}$$

are different and how they can be related. Here $y_j \simeq y(n\Delta t)$ is a discretization with $y_0 = 1$ while the z_j are uncorrelated Gaussian random numbers with unit variance,

(a) Show that in the Ito scheme the expectation value is $\langle y_j \rangle = 1$ for all j, meaning that the Ito scheme yields **wrong** results. (1P)

(b) Solve the Stratonovic update rule for y_{j+1} and argue why this update prescription could be numerically unstable. (1P)

(c) Let us ignore the instability problem discovered in (b) and let us Taylor-expand the right-hand side of the update rule obtained in (b) up to fourth order in $\sqrt{\Delta t}$ around $\Delta t = 0$. Show that in the limit $\Delta t \to 0$ the Stratonovic update gives the correct expectation value. (2P)

(d) Add an additional deterministic drift term of the form $+\lambda \Delta t$ in the Ito scheme such that it yields the correct result in the limit $\Delta t \to 0$. (1P)

SAMPLE SOLUTION

(a) First we write down the expectation value

$$\langle y_{j+1} \rangle = \big\langle y_j + \sqrt{\Delta t}\, y_h u_j \big\rangle = \langle y_j \rangle + \sqrt{\Delta t} \langle y_j z_j \rangle.$$

It is important to realize that z_j is a freshly drawn random number that has not been used before, hence z_j and y_j are uncorrelated. For this reason the expectation value on the right hand side factorizes and we get:

$$\langle y_{j+1} \rangle = \langle y_j \rangle + \sqrt{\Delta t} \langle y_j \rangle \underbrace{\langle z_j \rangle}_{=0}.$$

This implies $\langle y_{j+1} \rangle = \langle y_j \rangle$, hence the expectation value is conserved in time, in contradiction with the results $\langle y(t) \rangle \sim e^{t/2}$.

(b) Solving the Stratonovic update rule for y_{j+1} one gets

$$y_{j+1} = \frac{1 + \frac{1}{2}\sqrt{\Delta t} z_j}{1 - \frac{1}{2}\sqrt{\Delta t} z_j} y_j$$

with the same random number z_j in the nominator and denominator. This update rules diverges as soon as the denominator vanishes, i.e., if the random number exceeds $z_j > 2/(\Delta t)$. Fortunately such events are strongly (superexponentially) suppressed as $\Delta t \to 0$.

(c) The Taylor-expanded update rule reads:

$$y_{j+1} = \Big(1 + z_j\sqrt{\Delta t} + \frac{z_j^2 \Delta t}{2} + \frac{z_j^3 (\Delta t)^{3/2}}{4} + \frac{z_j^4 (\Delta t)^2}{8} + \dots\Big) y_j$$

Again the expectation values factorize, i.e.

$$\langle y_{j+1}\rangle = \Big(1 + \langle z_j\rangle\sqrt{\Delta t} + \frac{\langle z_j^2\rangle \Delta t}{2} + \frac{\langle z_j^3\rangle (\Delta t)^{3/2}}{4} + \frac{\langle z_j^4\rangle (\Delta t)^2}{8} + \dots\Big) \langle y_j\rangle$$

where the expectation values appearing in the bracket are simply the moments of the normal distribution with unit variance:

$$\langle z_j\rangle = 0\,, \qquad \langle z_j^2\rangle = 1\,, \qquad \langle z_j^3\rangle = 0\,, \qquad \langle z_j^4\rangle = 3\,.$$

Thus we end up with

$$\langle y_{j+1}\rangle = \Big(1 + \frac{\Delta t}{2} + \frac{3(\Delta t)^2}{8} + \dots\Big) \langle y_j\rangle$$

For $\Delta t \to 0$ only the linear term contributes, giving $y(t) \sim e^{t/2}$.

(d) Simply choose $\lambda = 1/2$.

Exercise 10.2: Schwinger-Dyson equation for DP (7P)

The Langevin equation of directed percolation (DP) reads:

$$\frac{\partial}{\partial t}\phi(\mathbf{x},t) = D\nabla^2\phi(\mathbf{x},t) + a\phi(\mathbf{x},t) - \frac{\lambda}{2}\phi(\mathbf{x},t)^2 + \xi(\mathbf{x},t)\,,$$

where $\xi(\mathbf{x},t)$ is a field-dependent white noise with the correlations

$$\langle \xi(\mathbf{x},t)\rangle = 0\,, \qquad \langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = 2\Gamma\phi(\mathbf{x},t)\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

The parameters D, a, λ, Γ are given constants. The parameter a is the critical parameter which has to be tuned to a particular value at the transition

(a) Determine the parameter a, the exponents z, χ, and the dimension d in such a way that the Langevin equation is invariant under scale transformations (1P)

$$x \to x' = bx\,, \quad t \to t' = b^z t\,, \quad \phi(\mathbf{x},t) \to \phi'(\mathbf{x}',t') = b^\chi \phi(\mathbf{x},t)\,.$$

(b) Suppose that the parameters D, a, λ, Γ have some given values. If one performs an infinitesimal scale transformation with a dilatation $b = 1+\epsilon \quad (|\epsilon| \ll 1)$, then the resulting Langevin equation can be considered as formally equivalent to the old one but with slightly different coefficients $D', a', \lambda', \Gamma'$. Determine these new coefficients as a function of the old ones to first order in ϵ. (1P)

(c) Apply the Fourier transformation

$$\phi(\mathbf{k},\omega) = \int \mathrm{d}^d x\ \mathrm{d}t\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\phi(\mathbf{x},t)\,, \qquad \phi(\mathbf{x},t) = \frac{1}{(2\pi)^{d+1}}\int \mathrm{d}^d k\ \mathrm{d}\omega\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\phi(\mathbf{k},\omega)$$

to the Langevin equation to derive a transformed equation in $\phi(\mathbf{k},\omega)$. (1P)
What are the correlations of the Fourier-transformed noise $\xi(\mathbf{k},\omega)$? (1P)

(d) Considering a field theory "at the tree level" means to drop the interaction term (ϕ^2 in our case or the corresponding convolution term in the Fourier-transformed equation). On the tree level the Langevin equation can be written as a linear differential equation on the left hand side and the noise term on the right hand side. Setting the noise term to zero, determine the bare (=tree level, denoted by an index 0) Greens function $G_0(\mathbf{k},\omega)$ and find a formal solution for $\phi_0(\mathbf{k},\omega)$. (1P)

(e) Inserting this result back into the Fourier-transformed Langevin equation show that one obtains an integral equation (1P)

$$\phi(\mathbf{k},\omega) = \phi_0(\mathbf{k},\omega) - \frac{\lambda}{2}G_0(\mathbf{k},\omega)\frac{1}{(2\pi)^{d+1}}\int \mathrm{d}^d k'\ \mathrm{d}\omega'\ \phi(k',\omega')\phi(\mathbf{k}-\mathbf{k}',\omega-\omega')$$

(f) Suggest an approximation for $\phi(\mathbf{k},\omega)$. (1P)

Sample Solution

(a) Under the replacement $\mathbf{x}\to b\mathbf{x}$, $t\to b^z t$, and $\phi\to b^\chi\phi$ the given Langevin equation turns into

$$\frac{\partial}{\partial t}\phi(\mathbf{x},t) = \underbrace{b^{z-2}D}_{D'}\nabla^2\phi(\mathbf{x},t) + \underbrace{b^z a}_{a'}\phi(\mathbf{x},t) - \frac{1}{2}\underbrace{b^{\chi+z}\lambda}_{\lambda'}\phi^2(\mathbf{x},t) + \underbrace{b^{\frac{-\chi-d+z}{2}}\zeta(\mathbf{x},t)}_{\zeta'(\mathbf{x},t)}$$

where we have already divided by the factor $b^{\chi-z}$ coming from the left side of the equation. Obviously, the equation is invariant under rescaling (meaning that $D=D'$ etc.) if all b-powers are neutral. The first term gives $z=2$. Then the second term would be relevant ($\sim b^2$), hence the only way to establish scale invariance is to take $a=0$. This is nothing but the critical point in mean field. Furthermore we get $\chi=-2$. As for the noise, we note that

$$\langle\xi'(\mathbf{x},t)\xi'(\mathbf{x}',t')\rangle = \underbrace{\Gamma\, b^{\chi-d-z}}_{\Gamma'}\phi(\mathbf{x},t)\,\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')\,.$$

To summarize, scale invariance in the mean field limit can be established if

$$a=0 \qquad z=2\,, \qquad \chi=-2\,, \qquad d\geq d_c=4.$$

Here $d_c=4$ is the upper critical dimension below which fluctuation corrections become relevant.

Remark: The exponents given above are related to the three standard exponents $\beta,\nu_\parallel,\nu_\perp$ by

$$\chi=-\beta/\nu_\perp\,, \qquad z=\nu_\parallel/\nu_\perp\,,$$

giving the mean field exponents

$$\beta=1\,, \qquad \nu_\perp=\frac{1}{2}\,, \qquad \nu_\parallel=1.$$

(b) For an infinitesimal dilatation with $b = 1 + \epsilon$ the RG flow equation take the form

$$\begin{aligned} D' &= (1+\epsilon(z-2))D \\ a' &= (1+\epsilon z)a \\ \lambda' &= (1+\epsilon(\chi+z))\lambda \\ \Gamma' &= (1+\epsilon(z-\chi-d))\Gamma. \end{aligned}$$

As one can see, if one chooses the mean field exponents, this RG flow equation has a fixed point provided that a is at the critical point $a_c = 0$.

(c) By Fourier transformation it is straight-forward to shoe that the Langevin equation turns into

$$\begin{aligned} i\omega\phi(\mathbf{k},\omega) &= -Dk^2\phi(\mathbf{k},\omega) + \xi(\mathbf{k},\omega) + a\,\phi(\mathbf{k},\omega) \\ &\quad -\frac{\lambda}{2}\frac{1}{(2\pi)^{d+1}}\int_{-\infty}^{+\infty} \mathrm{d}^d k\ \mathrm{d}\omega\,\phi(\mathbf{k}',\omega')\,\phi(\mathbf{k}-\mathbf{k}',\omega-\omega') \end{aligned}$$

where the Fourier-transformed noise now takes the correlations

$$\langle \xi(\mathbf{k},\omega)\xi(\mathbf{k}',\omega')\rangle = \Gamma\,\phi(\mathbf{k}+\mathbf{k}',\omega+\omega')\,.$$

(d) Tree level means to drop the convolution product (the integral) in the expression given above. Without the noise the equation reads

$$(-i\omega - a + Dk^2)G_0(\mathbf{k},\omega) = 1/\sqrt{2\pi},$$

where the constant 1 on the rhs. just represents the Fourier-transformed delta function. The free solution is therefore given by

$$G_0(\mathbf{k},\omega) = \frac{1}{Dk^2 - a - i\omega}\,, \qquad \phi_0(\mathbf{k},\omega) = G_0(\mathbf{k},\omega)\xi(\mathbf{k},\omega).$$

(e) Inserting this free solution back into the integral equation it can be written in the form

$$\phi(\mathbf{k},\omega) = \phi_0(\mathbf{k},\omega) - \frac{\lambda}{2}G_0(\mathbf{k},\omega)\frac{1}{(2\pi)^{d+1}}\int_{-\infty}^{+\infty} \mathrm{d}^d k\,\mathrm{d}\omega\,\phi(\mathbf{k},\omega)\,\phi(\mathbf{k}-\mathbf{k}',\omega-\omega')\,.$$

(f) This integral equation does not yet solve the problem in practice since the solution on the lhs. appears also quadratically on the rhs. However, it suggests the approximation scheme to first insert the free (tree-level) solution on the r.h.s and then to obtain an improved approximation on the lhs:

$$\phi(\mathbf{k},\omega) \approx \phi_0(\mathbf{k},\omega) - \frac{\lambda}{2}G_0(\mathbf{k},\omega)\frac{1}{(2\pi)^{d+1}}\int_{-\infty}^{+\infty} \mathrm{d}^d k\,\mathrm{d}\omega\,\phi_0(\mathbf{k},\omega)\,\phi_0(\mathbf{k}-\mathbf{k}',\omega-\omega')\,.$$

This is the basis of a so-called one-loop expansion.

(Σ = 12P)