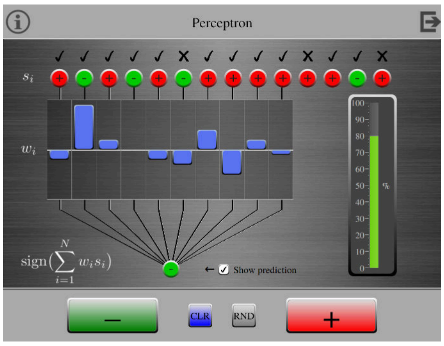
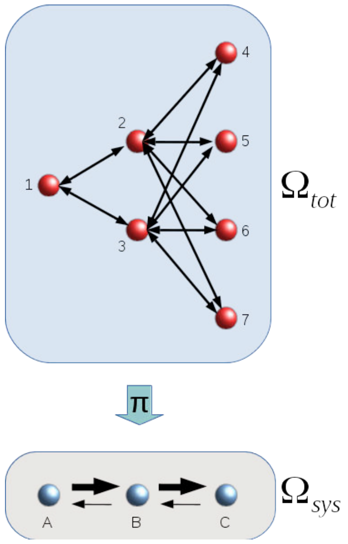



Try our app **Perceptron**, available at Google Play

## Exercise 12.1: Entropy production (8P)




Consider a closed 'total' system with seven configurations, as shown in the figure. The configurations are connected by arrows which represent symmetric rates $w_{c\to c'} = w_{c'\to c} = 1$, while missing arrows stand for zero rates.

Furthermore, consider an embedded subsystem labeled by 'sys' with three configurations $\{A, B, C\}$ corresponding to (1), (2,3), and (4,5,6,7) in the total system.

(a) Find the Liouvillian $\mathcal{L}_{\text{tot}}$ for the total system. (1P)

(b) Suppose that the system is initially in the leftmost configuration $c = 1$ at $t = 0$. Compute the time-dependent probability distribution $|P_{\text{tot}}(t)\rangle$ by solving the master equation. Verify that the system relaxes towards an equally distributed stationary state. (2P)

(c) Determine the coarse-grained probability distribution $|P_{\text{sys}}(t)\rangle$ and compute the Shannon entropy $S_{\text{sys}}(t)$. Approximate the time at which $S_{\text{sys}}(t)$ is maximal. (1P)

(d) Determine the effective rates $w_{s\to s'}(t)$ in the subsystem and confirm that they are *not* time-dependent in the present example. (1P)

(e) The entropy production in the environment is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{\text{env}}(t) = \sum_{s,s' \in \Omega_{\text{sys}}} J_{s\to s'}(t) \, \ln \frac{w_{s\to s'}(t)}{w_{s'\to s}(t)} \, .$$

Compute and integrate this expression to get an exact expression for the entropy in the environment as a function of time. (2P)

(f) Plot $S_{\text{sys}}(t)$, $S_{\text{env}}(t)$, and $S_{\text{tot}}(t)$ in a single plot in the range $t \in [0, 3]$ and confirm that $S_{\text{tot}}(t) = S_{\text{sys}}(t) + S_{\text{env}}(t)$. (1P)

*Please turn over* ⇒

## Exercise 12.2: The perceptron **(4P)**

The perceptron is used as a simple building block of artificial neural networks. Please read about the perceptron in the literature (e.g. https://en.wikipedia.org/wiki/Perceptron). The purpose of this exercise is to demonstrate that the perceptron can be used to implement certain Boolean functions. More specifically, if 'true' and 'false' are represented by +1 and -1, the perceptron function $s^{out} = \text{sign}(\vec{J} \cdot \vec{s}^{\,in} - \theta)$ with suitable weights $\vec{J}$ can be used to represent a Boolean function.

(a) Find a set of weights $\vec{J}$ and a threshold $\theta$ such that a perceptron with two inputs and a single output reproduces the NOR function defined by the following table: (1P)

| | | | | |
|---|---|---|---|---|
| $s_1^{in}$ | -1 | 1 | -1 | 1 |
| $s_2^{in}$ | -1 | -1 | 1 | 1 |
| $s^{out}$ | 1 | -1 | -1 | -1 |

(b) Prove that it is impossible to represent the XOR function

| | | | | |
|---|---|---|---|---|
| $s_1^{in}$ | -1 | 1 | -1 | 1 |
| $s_2^{in}$ | -1 | -1 | 1 | 1 |
| $s^{out}$ | -1 | 1 | 1 | -1 |

by a single perceptron with two inputs and one output. (1P)

(c) Show that the Boolean function

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $s_1^{in}$ | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 |
| $s_2^{in}$ | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 1 |
| $s_3^{in}$ | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 |
| $s^{out}$ | -1 | 1 | -1 | -1 | 1 | 1 | -1 | 1 |

can be represented by a single perceptron with three inputs and one output. (2P)
*Hint*: Read about the learning rule for a perceptron.

**($\Sigma$ = 12P)**

**Vielen Dank für Ihr Engagement und Ihre Mitarbeit in diesem Corona-Semester.**

To be handed in electronically until Wednesday, July 15, 2020, 12:00, on WueCampus according to our Corona guidelines on the web page `cs.hayehinrichsen.de`.