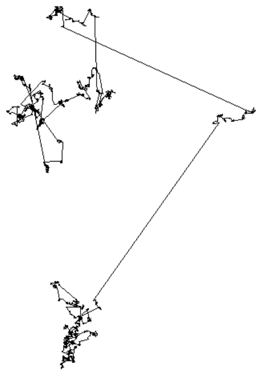

Levy flight of a single particle

Exercise 11.1: Levy-stable distributions (12P)

Definition: A probability density $P_X(x)$ is called *stable* if the sum of uncorrelated random variables distributed according to P_X is (up to rescaling) again distributed according to P_X. The best-known example is a normal distribution: The sum of two normally distributed random variables is again normally distributed.

(a) Let X_1 and X_2 be two uncorrelated random variables distributed according to $P_X(x)$. Consider the sum $Y = a_1X_1 + a_2X_2$, where a_1, a_2 are constants. Express the probability density $P_Y(y)$ of Y as a convolution product. (2P)

(b) According to the definition given above, P_X is stable if

$$P_Y(y) = \frac{1}{R(a_1, a_2)} P_X\Big(\frac{y}{R(a_1, a_2)}\Big), \tag{1}$$

where $R(a_1, a_2)$ is a scaling factor depending only on the constants $a_1, a_2 > 0$. Prove that this relation preserves normalization. (2P)

(c) Apply a Fourier transformation and express the convolution product obtained in (a) as an ordinary product in k-space. (2P)

(d) Using this result, show that the stability condition (1) expressed in Fourier space reads

$$\tilde{P}_X(a_1k)\tilde{P}_X(a_2k) = \frac{1}{\sqrt{2\pi}}\tilde{P}_X\Big(R(a_1, a_2)k\Big),$$

where $\tilde{P}_X(k)$ is the Fourier-transform of $P_X(x)$. (2P)

(e) Show that the solution of the stability condition is of the form

$$\tilde{P}_X(k) = \begin{cases} B_+ \exp(-(+k)^\alpha) & \text{for } k > 0 \\ 1/\sqrt{2\pi} & \text{for } k = 0 \\ B_- \exp(-(-k)^\alpha) & \text{for } k < 0 \end{cases}$$

with a parameter $\alpha > 0$, called *Lévy-stable distributions.* (1P)

Please turn over ⇒

(f) To obtain $P_X(x)$, one still has to perform the inverse Fourier transform back to real space. Unfortunately, this is generally impossible in a closed form. However, it is possible to carry out this inverse Fourier transformation numerically, e.g. with *Mathematica*® . Try to do this and plot the result qualitatively for the symmetric case $B_+ = B_-$ and for the three values $\alpha = 1.5, 2.0$, and 2.5. (1P)

(g) Why is your result for $\alpha = 2.5$ unphysical? What does it mean in practice? (1P)

(Σ = 12P)

To be handed in electronically until Wednesday, July 08, 2020, 12:00, on WueCampus according to our Corona guidelines on the web page `cs.hayehinrichsen.de`.