SAMPLE SOLUTIONS EXERCISE 8

EXERCISE 8.1: 1D ISING MODEL WITH FREE BOUNDARY CONDITIONS (10P)

Let us consider the one-dimensional Ising model of N classical spins $s_n = \pm 1$ with free (non-periodic) boundary conditions

$$E_N = -\sum_{n=1}^{N-1} s_n s_{n+1}.$$

(a) Compute the partition sum $Z_N = \sum_{\{s\}} e^{-\beta E_N}$ recursively. (2P)

(b) Compute the free energy F, the entropy H, the internal energy U, and the heat capacity $C = T\frac{\partial H}{\partial T}$ (you may set $k_B = 1$). (2P)

(c) Prove that $\langle s_j \rangle = 0$. (1P)

(d) Show that the connected spin correlation function $G_{ij} = \langle s_i s_j \rangle - \langle s_i \rangle\langle s_j \rangle$ decays exponentially with increasing distance $r = |j - i|$ as $G_{ij} \sim e^{-r/\xi}$. Show that the correlation length is given by $\xi = -\ln(\tanh\beta)]^{-1}$. Interpret the result in the limit $T \to 0$. *Hint:* $\langle s_i s_j \rangle = \langle s_i s_{i+1} s_{i+1} s_{i+2} s_{i+2} \ldots s_{j-1} s_{j-1} s_j \rangle$ (3P)

(e) Calculate the magnetic susceptibility $\chi(\beta)$ in zero magnetic field $h = 0$ by using the so-called fluctuation-dissipation theorem

$$\frac{\chi(\beta)}{N} = \beta \sum_{n=1}^{N} G_{\frac{N}{2}n}$$

in the thermodynamic limit $N \to \infty$. For simplicity assume N to be even. What happens for $T \to 0$? (2P)

SAMPLE SOLUTION

(a) We anchor the recursion at the lowest possible value $N = 2$:

$$Z_2(\beta) = \sum_{s_1=\pm 1}\sum_{s_2=\pm 1} e^{\beta s_1 s_2} = 2\sum_{s=\pm 1} e^{\beta s} = 4\cosh\beta$$

For the recursion we exploit that $E_N = E_{N-1} - s_{N-1}s_N$ to show that

$$Z_N(\beta) = \sum_{s_1=\pm 1}\ldots\sum_{s_{N-1}=\pm 1} e^{-\beta E_{N-1}} \sum_{s_N=\pm 1} e^{\beta s_{N-1}s_N} = Z_{N-1}\sum_{s=\pm 1} e^{\beta s},$$

hence we get the recursion $Z_N = Z_{N-1} 2\cos\beta$ with the solution

$$\boxed{Z_N(\beta) = 2^N(\cosh\beta)^{N-1}}$$

(b) The results are:

- Free energy:

$$F \;=\; -\frac{1}{\beta}\ln\big[Z_N(\beta)\big] \;=\; -\frac{1}{\beta}N\ln 2 - \frac{1}{\beta}(N-1)\ln\big[\cosh\beta\big].$$

- Entropy:

$$H \;=\; \frac{1}{\beta^2}\frac{\partial F}{\partial\beta} \;=\; N\ln 2 + (N-1)\ln\big[\cosh\beta\big] - (N-1)\beta\tanh\beta\,.$$

- Internal energy:

$$U \;=\; -\frac{\partial \ln Z_N(\beta)}{\partial\beta} \;=\; -(N-1)\tanh\beta$$

- Heat capacity:

$$C \;=\; -\beta\frac{\partial H}{\partial\beta} \;=\; \frac{(N-1)\beta^2}{\cosh^2(\beta)}$$

(c) The expectation value reads

$$\langle s_j\rangle = \frac{1}{Z_N}\sum_{s_1\ldots s_N=\pm1} s_j e^{\beta\sum_{n=1}^{N-1}s_n s_{n+1}}$$

For each configuration $s = s_1,\ldots,s_N$ there is a Z_2-flipped configuration $-s = -s_1,\ldots,-s_N$ in the sum. This flipped configuration has the same energy (because the energy is quadratic in the spins) but the s_j changes sign. This means that all summands of the sum cancel pairwise.

(d) Because of $\langle s_j\rangle = 0$ the connected and the bare correlator are identical, i.e. we have $G_{ij} = \langle s_i s_j\rangle$. To compute the correlation function we consider a modified energy

$$E_N = -\sum_{n=1}^{N-1} J_n s_n s_{n+1},$$

where we will finally set all $J_n = 1$. The corresponding partition sum reads

$$Z_N(\beta, J_1,\ldots,J_{N-1}) \;=\; 2^N\prod_{n=1}^{N-1}(\cosh\beta J_n).$$

Moreover, we use a trick and represent the correlator as

$$\langle s_i s_j\rangle \;=\; \langle s_i s_{i+1}s_{i+1}s_{i+2}s_{i+2}\ldots s_{j-1}s_{j-1}s_j\rangle$$

Now we can compute

$$\langle s_i s_j\rangle \;=\; \frac{1}{Z_N}\sum_s (s_i s_{i+1})(s_{i+1}s_{i+2})\cdots(s_{j-1}s_j)\exp\Big(\sum_{n=1}^{N-1}J_n s_n s_{n+1}\Big)$$

and this can be expressed through derivatives:

$$\langle s_i s_j\rangle \;=\; \frac{1}{Z_N}\frac{\partial^{j-i}}{\partial J_i\partial J_{i+1}\cdots\partial J_{j-1}}Z_N(\beta, J_1,\ldots,J_{N-1})\Big|_{J_1\ldots J_{N-1}=1}$$

$$\Rightarrow\quad \langle s_i s_j\rangle \;=\; (\tanh\beta)^{|j-i|}.$$

Comparing this with $\langle s_i s_j \rangle \sim e^{-|j-i|/\xi}$ we get the correlation length

$$\boxed{\xi = -\frac{1}{\ln(\tanh\beta)} > 0.}$$

which diverges for $T \to 0$ ($\beta \to \infty$)

(e) In the thermodynamic limit we can write

$$\lim_{N\to\infty} \sum_{n=1}^{N} G_{\frac{N}{2}n} = \sum_{d=-\infty}^{\infty} (\tanh\beta)^{|d|} = 1 + \sum_{d=1}^{\infty} (\tanh\beta)^d = 1 - \frac{2\tanh\beta}{\tanh\beta - 1} = e^{2\beta}.$$

$$\Rightarrow \quad \chi(\beta) = N\beta e^{2\beta}.$$

For $T \to 0$ ($\beta \to \infty$) the susceptibility diverges exponentially.

Exercise 8.2: Bogoliubov transformation (2P)

A Bogoliubov transformation is a canonical transformation that mixes creation and annihilation operators without destroying the commutation relations.

Let $\psi_1, \psi_1^\dagger$ and $\psi_2, \psi_2^\dagger$ be two pairs of fermionic operators, that is $\{\psi_i^\dagger, \psi_j^\dagger\} = \{\psi_i, \psi_j\} = 0$ and $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$. Find coefficients $u, v \in \mathbb{C}$ such that

$$\eta = u\psi_1 + v\psi_2^\dagger, \qquad \eta^\dagger = u^*\psi_1^\dagger + v^*\psi_2$$

obey fermionic anticommutation relations and find a general parametrization of u and v.

Sample Solution

We first check that

$$\{\eta, \eta\} = u^2\{\psi_1, \psi_1\} + uv\{\psi_1, \psi_2^\dagger\} + uv\{\psi_2^\dagger, \psi_1\} + v^2\{\psi_2^\dagger, \psi_2^\dagger\} = 0$$

since all anticommutators on the r.h.s vahish. Similarly we can show that $\{\eta^\dagger, \eta^\dagger\} = 0$. For the remaining anticommutator we get

$$\{\eta, \eta^\dagger\} = uu^*\{\psi_1, \psi_1^\dagger\} + uv^*\{\psi_1, \psi_2\} + vu^*\{\psi_2^\dagger, \psi_1^\dagger\} + vv^*\{\psi_2^\dagger, \psi_2\} = uu^* + vv^* = |u|^2 + |v|^2.$$

This can be parametrized by

$$u = e^{i\phi_1}\cos\theta, \qquad v = e^{i\phi_2}\sin\theta.$$

(Σ = 12P)