## Sample Solutions Exercise 7

### Exercise 7.1: The first gap (6P)

The Hamiltonian of the Ising quantum chain is defined by

$$\mathbf{H}_N = \sum_{n=0}^{N-1} \Big( \sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^z \Big)$$

where $\lambda > 0$ and $\sigma_n^{x|y|z} = \mathbb{1}_{2^n \times 2^n} \otimes \sigma^{x|y|z} \otimes \mathbb{1}_{2^{N-n-1} \times 2^{N-n-1}}$ are Pauli matrices acting at site $n$, assuming periodic boundary conditions. The first (or lowest) gap $\Delta_N$ is defined as the difference between the first two lowest-lying eigenvalues of $\mathbf{H}_N$.

(a) In Markov processes, the first gap of $\mathcal{L}$ is the inverse leading relaxation time. In particle physics, the first gap measures the mass. What is the interpretation of the first gap of $\mathbf{H}_N$ in the present context? (1P)

(b) With the *Mathematica*® code fragment for the tensor product $\otimes$ given in the lecture notes,[1] write a code snippet to set up the Hamiltonian for arbitrary $N$ and $\lambda$. (2P)
Cross check: The eigenvalues for $N = 3$ (i.e. 8×8) are $\big\{-\lambda - 1, -\lambda - 1, \lambda - 1, \lambda - 1, -2\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1, 2\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1, -2\sqrt{\lambda^2 + \lambda + 1} - \lambda + 1, 2\sqrt{\lambda^2 + \lambda + 1} - \lambda + 1\big\}$

(c) With (b) compute *numerically*[2] the first gap of $\mathbf{H}_N$ for $N = 2, 4, 6, 8, 10, 12$ and and plot the gap for $\lambda = 0.8, 1, 1.2$ double-logarithmically as a function of $N$. For which $\lambda$ do you get an almost straight line of points and why?[3] (2P)

(d) Compute the negative *discrete logarithmic derivative*

$$\nu(N) = -\frac{\ln[\Delta_{N+2}/\Delta_N]}{\ln[(N+2)/N]}$$

for $\lambda = 1$ and $N = 2, 4, 6, 8, 10$ and guess the value of the critical exponent $\nu$ which is defined as the limit $\nu = \lim_{N\to\infty} \nu(N)$. (1P)

### Sample Solution

(a) In the present case the Hamiltonian $\mathbf{H}$ is related to the transfer matrix $\mathbf{T} = e^{\mathbf{H}}$, hence the first gap can be interpreted as the *correlation length* in vertical direction. (1P)

(b) A possible solution reads:

```
 Attributes[CircleTimes] = {Flat, OneIdentity};
CircleTimes[a_List /; VectorQ[a], b_List /; VectorQ[b]] :=
  Flatten[KroneckerProduct[a, b]];
```

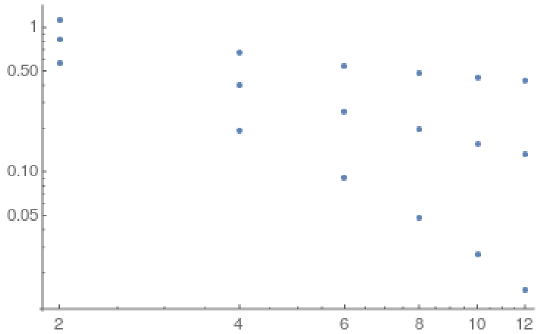
[1] You are of course free to use any other algebraic computer systems as well.
[2] In Mathmatica: Use `N[...]` to make $\mathbf{H}$ numerical: `Eigenvalues[N[H[length,lambda]]]`
[3] The execution for $N = 12$ should take less than 20 seconds. If not, go only up to $N = 10$.

```
CircleTimes[a_List /; MatrixQ[a], b_List /; MatrixQ[b]] :=
  KroneckerProduct[a, b];
  sx = {{0, 1}, {1, 0}};
sy = {{0, -I}, {I, 0}};
sz = {{1, 0}, {0, -1}};
Id[n_] := IdentityMatrix[n];

 H[L_, \[Lambda]_] := Sum[
   Id[2^j]\[CircleTimes]sx\[CircleTimes]sx\[CircleTimes]Id[2^(
     L - j - 2)], {j, 0, L - 2}] +
  sx\[CircleTimes]Id[2^(L - 2)]\[CircleTimes]sx +
  \[Lambda]*
   Sum[Id[2^j]\[CircleTimes]sz\[CircleTimes]Id[2^(L - j - 1)], {j, 0,
     L - 1}]
```

(c) To extract the gap you can use e.g. the following function:

```
 gap[L_, \[Lambda]_] := Module[{spec},
  spec = H[L, \[Lambda]] // N // Eigenvalues // Sort;
  spec[[2]] - spec[[1]]
  ]
```




Only the points for $\lambda = 1$ are almost straight. The reason is that $\lambda_c = 1$ is the critical point (see lecture notes), and at the critical point we expect power laws which give straight lines in double-logarithmic representations.

(d) The logarithmic derivative gives

$$\{1.05824, 1.01784, 1.00874, 1.0052, 1.00345\}$$

and we would conjecture that $\nu = 1$. If you wish to do it more carefully, you may e.g. plot these values against $1/N^2$, which gives again an almost straight line that you can extrapolate to this value. $\nu = 1$ is in fact the correct result for the Ising correlation length exponent in 2D (= 1D quantum).

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⇒ PLEASE TURN OVER

## Exercise 7.2: XXZ chain and 2-point scalar operators (6P)

The $XXZ$-Heisenberg chain with $N$ sites and open boundary conditions is defined by the Hamiltonian

$$H = -\sum_{j=1}^{N-1} e_j\,, \qquad e_j = -\frac{1}{2}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z - \mathbb{1}\Big)\,.$$

In this exercise we want to improve algebraic skills. Please do not construct explicit matrices via *Mathematica*® , but use commutation relations instead. Of course, you may use *Mathematica*® to verify your results.

(a) Show *algebraically* that the $e_j$ satisfy the Temperley-Lieb algebra (c.f. 3.2) (2P)

$$e_j^2 = 2e_j\,, \qquad e_j e_{j\pm1} e_j = e_j\,, \qquad [e_i, e_j] = 0 \text{ for } |i-j| \geq 2$$

Hint: The Pauli matrices $(\sigma^x, \sigma^y, \sigma^z) = (\sigma^1, \sigma^2, \sigma^3)$ obey the multiplication rule $\sigma^\mu\sigma^\nu = \delta_{\mu\nu}\mathbb{1} + i\sum_{\tau=1}^{3}\epsilon_{\mu\nu\tau}\sigma^\tau$.

(b) Verify *algebraically* that the $SU(2)$ operators

$$S^\pm = \sum_{j=1}^{N}\sigma^\pm\,, \qquad S^z = \frac{1}{2}\sum_{j=1}^{N}\sigma^z \qquad \sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$$

obey the commutation relations $[S^+, S^-] = 2S^z$ and that they commute with the operators $e_1, \ldots, e_{N-1}$. (2P)

(c) The two-point scalar operator is defined as

$$C_{l,m} := \frac{1}{2}\Big(\mathbb{1} - \vec{\sigma}_l \cdot \vec{\sigma}_m\Big)$$

Prove that (1P)

$$C_{l,m} = (\mathbb{1} - e_{m-1})(\mathbb{1} - e_{m-2})\cdots(\mathbb{1} - e_{l+1})\, e_l\, (\mathbb{1} - e_{l+1})(\mathbb{1} - e_{l+2})\cdots(\mathbb{1} - e_{m-1})\,,$$

(implying via (b) that the two-point operator $C_{m,n}$ is SU(2)-invariant) and show that the correlator obeys the recurrence relation ($l<m<n$) (1P)

$$C_{l,n} = C_{l,m} + C_{m,n} - C_{l,m}C_{m,n} - C_{m,n}C_{l,m}.$$

### Sample Solution

In the following the sum over greek indices runs over $(1,2,3) = (x,y,z)$.

(a) First we prove the first relation $e_j^2 = 2e_j$:

$$\begin{aligned} e_j^2 &= \frac{1}{4}\Big(\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu - \mathbb{1}\Big)^2 \\ &= \frac{1}{4}\Big(\sum_{\mu\nu}\sigma_j^\mu\sigma_{j+1}^\mu\sigma_j^\nu\sigma_{j+1}^\nu - 2\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu + \mathbb{1}\Big) \\ &= \frac{1}{4}\Big(\sum_{\mu\nu}(\sigma_j^\mu\sigma_j^\nu)(\sigma_{j+1}^\mu\sigma_{j+1}^\nu) - 2\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu + \mathbb{1}\Big) \end{aligned}$$

Now we insert the relation $\sigma^\mu\sigma^\nu = \delta_{\mu\nu} + i\sum_\rho \epsilon_{\mu\nu\rho}\sigma^\rho$. Note that in contrast to Special Relativity, we do not care about lower and upper indices here:

$$\begin{aligned}
e_j^2 &= \frac{1}{4}\Big(\sum_{\mu\nu}\big(\delta_{\mu\nu}\mathbb{1} + i\sum_\rho \epsilon_{\mu\nu\rho}\sigma_j^\rho\big)\big(\delta_{\mu\nu}\mathbb{1} + i\sum_\tau \epsilon_{\mu\nu\tau}\sigma_{j+1}^\tau\big) - 2\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu + \mathbb{1}\Big) \\
&= \frac{1}{4}\Big(\underbrace{\big(\sum_{\mu\nu}\delta_{\mu\nu}\mathbb{1}\big)}_{=3\mathbb{1}} + 0 + 0 - \sum_{\rho\tau}\underbrace{\sum_{\mu\nu}\epsilon_{\mu\nu\rho}\epsilon_{\mu\nu\tau}}_{=2\delta_{\rho\tau}}\sigma_j^\rho\sigma_{j+1}^\tau - 2\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu + \mathbb{1}\Big) \\
&= \frac{1}{4}\Big(4\mathbb{1} - 2\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu - 2\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu\Big) \;=\; 2e_j \qquad \square
\end{aligned}$$

Then we prove the second relation $e_je_{j\pm1}e_j = e_j$. For simplicity we take only the positive sign (the proof for the negative sign is fully analogous):

$$\begin{aligned}
e_je_{j+1}e_j &= -\frac{1}{2}\Big(e_j\big(\sum_\nu \sigma_{j+1}^\nu\sigma_{j+2}^\nu - \mathbb{1}\big)e_j\Big) \\
&= -\frac{1}{2}e_j\big(\sum_\nu \sigma_{j+1}^\nu\sigma_{j+2}^\nu\big)e_j + \frac{1}{2}e_j^2 \\
&= -\frac{1}{2}\sum_\nu \big(e_j\sigma_{j+1}^\nu e_j\big)\,\sigma_{j+2}^\nu \,+\, e_j
\end{aligned}$$

Obviously, in order to prove $e_je_{j\pm1}e_j = e_j$, we have to prove that the first term in the lowest line vanishes. To this end it is sufficient to prove that the term in the round bracket $e_j\sigma_{j+1}^\nu e_j$ vanishes for all $\nu = 1,2,3$. This can be done as follows. We compute the anticommutator

$$\begin{aligned}
\{e_j, \sigma_{j+1}^\nu\} &= -\frac{1}{2}\Big\{\Big(\sum_\mu \sigma_j^\mu\sigma_{j+1}^\mu - \mathbb{1}\Big), \sigma_{j+1}^\nu\Big\} \\
&= -\frac{1}{2}\Big(\sum_\mu \sigma_j^\mu\underbrace{\{\sigma_{j+1}^\mu, \sigma_{j+1}^\nu\}}_{=2\delta_{\mu\nu}} - \underbrace{\{\mathbb{1}, \sigma_{j+1}^\nu\}}_{=\sigma_{j+1}^\nu}\Big) \;=\; \sigma_{j+1}^\mu - \sigma_j^\mu
\end{aligned}$$

and likewise $\{e_j, \sigma_j^\nu\} = -\sigma_{j+1}^\mu + \sigma_j^\mu$. Multiplying these relations from the left and from the right with $e_j$ one finds that $3e_j\sigma_{j+1}^\nu e_j = -e_j\sigma_j^\nu e_j$ and $3e_j\sigma_j^\nu e_j = -e_j\sigma_{j+1}^\nu e_j$, implying that $e_j\sigma_{j+1}^\nu e_j = 0$. This completes the proof.

**Remark:** There are many other ways to prove these algebraic identities. Here are some helpful general relations:

$$\{\sigma^\mu, \sigma^\nu\} = 2\delta_{\mu\nu}\,, \quad \sigma^\mu\sigma^\nu = \delta_{\mu\nu}\mathbb{1} + \sum_\rho \epsilon_{\mu\nu\rho}\sigma^\rho\,, \quad \sigma^\mu\sigma^\nu\sigma^\rho = \delta_{\mu\nu}\sigma^\rho + \delta_{\nu\rho}\sigma^\mu + \delta_{\rho\mu}\sigma^\nu$$

$$\sum_\mu \epsilon_{\mu\nu\rho}\epsilon_{\mu\tau\kappa} = \delta_{\nu\tau}\delta_{\rho\kappa} - \delta_{\nu\kappa}\delta_{\rho\tau}\,, \quad \sum_{\mu\nu}\epsilon_{\mu\nu\rho}\epsilon_{\mu\nu\tau} = 2\delta_{\rho\tau}\,, \quad \sum_{\mu\nu\rho}\epsilon_{\mu\nu\rho}\epsilon_{\mu\nu\rho} = 6$$

(b) The first relation is super-simple to prove:

$$[S^+, S^-] = \sum_{j,k=1}^N \underbrace{[\sigma_j^+, \sigma_k^-]}_{=\delta_{jk}\sigma^z} = \sum_j \sigma_j^z = 2S^0 \qquad \square$$

For the second relation we first compute the commutators ($\nu = 1, 2, 3,$):

$$[\sigma_j^\nu, e_j] = -\frac{1}{2}\Big(\sum_\mu [\sigma_j^\nu, \sigma_j^\mu \sigma_{j+1}^\mu] - \underbrace{[\sigma_j^\nu, \mathbb{1}]}_{=0}\Big)$$

$$= -\frac{1}{2}\sum_\mu \underbrace{[\sigma_j^\nu, \sigma_j^\mu]}_{=2i\sum_\rho \epsilon_{\nu\mu\rho}\sigma_j^\rho} \sigma_{j+1}^\mu = -i\sum_{\mu\rho} \epsilon_{\nu\mu\rho}\sigma_j^\rho \sigma_{j+1}^\mu$$

and

$$[\sigma_{j+1}^\nu, e_j] = -\frac{1}{2}\Big(\sum_\mu [\sigma_{j+1}^\nu, \sigma_j^\mu \sigma_{j+1}^\mu] - \underbrace{[\sigma_{j+1}^\nu, \mathbb{1}]}_{=0}\Big)$$

$$= -\frac{1}{2}\sum_\mu \sigma_j^\mu \underbrace{[\sigma_{j+1}^\nu, \sigma_{j+1}^\mu]}_{=2i\sum_\rho \epsilon_{\nu\mu\rho}\sigma_{j+1}^\rho} = -i\sum_{\mu\rho} \epsilon_{\nu\mu\rho}\sigma_j^\mu \sigma_{j+1}^\rho$$

As can be seen, both expressions have just the opposite sign, i.e. $[\sigma_j^\nu, e_j] = -[\sigma_{j+1}^\nu, e_j]$. The same hold for $\sigma^\pm$:

$$[\sigma_j^\pm, e_j] = -[\sigma_{j+1}^\pm, e_j].$$

All other commutators vanish:

$$[\sigma_k^\pm, e_j] = 0 \qquad \text{for } k \neq j \wedge k \neq j+1.$$

Since $[\sigma_j^\pm + \sigma_{j+1}^\pm, e_j] = 0$ it is clear that $[S^\pm, e_j] = 0$, implying that $[S^\pm, H] = 0$. Since $S^0 = \frac{1}{2}[S^+, S^-]$, this holds also for $S^0$, i.e., $[S^0, e_j] = [S^0, H] = 0$. □

(c) We can prove this relation recursively. The relation is anchored at

$$C_{l,l+1} = \frac{1}{2}\Big(\mathbb{1} - \sum_\mu \sigma_l^\mu \sigma_{l+1}^\mu\Big) = e_l$$

Now suppose that the given formula is correct for $C_{l,m-1}$. Then, by induction, we would like to show that it is also correct for $C_{l,m}$, which amounts to showing that

$$C_{l,m} = (\mathbb{1} - e_{m-1})C_{l,m-1}(\mathbb{1} - e_{m-1}),$$

meaning that

$$\big(\mathbb{1} - \sum_\mu \sigma_l^\mu \sigma_m^\mu\big) = \big(\mathbb{1} - e_{m-1}\big)\big(\mathbb{1} - \sum_\nu \sigma_l^\nu \sigma_{m-1}^\nu\big)\big(\mathbb{1} - e_{m-1}\big)$$

Because of $(\mathbb{1} - e_{m-1})^2 = \mathbb{1}$ we only have to show that

$$\sum_\mu \sigma_l^\mu \sigma_m^\mu = \big(\mathbb{1} - e_{m-1}\big)\Big(\sum_\nu \sigma_l^\nu \sigma_{m-1}^\nu\Big)\big(\mathbb{1} - e_{m-1}\big).$$

Since $l < m-1$ the left Pauli matrix $\sigma_l^\nu$ commutes with $e_{m-1}$, hence the condition reads

$$\sum_\mu \sigma_l^\mu \sigma_m^\mu = \sum_\nu \sigma_l^\nu \Big[\big(\mathbb{1} - e_{m-1}\big)\sigma_{m-1}^\nu \big(\mathbb{1} - e_{m-1}\big)\Big]$$

$$= \sum_\nu \sigma_l^\nu \Big[\sigma_{m-1}^\nu - \underbrace{\{e_{m-1}, \sigma_{m-1}^\nu\}}_{=\sigma_{m-1}^\nu - \sigma_m^\mu} + \underbrace{e_{m-1}\sigma_{m-1}^\nu e_{m-1}}_{=0}\Big] = \sum_\nu \sigma_l^\nu \sigma_m^\nu \qquad □$$

where we used the results from the previous part (b) under the curly brackets. For the recursion relation $C_{l,n} = C_{l,m} + C_{m,n} - C_{l,m}C_{m,n} - C_{m,n}C_{l,m}$ we first compute the last two terms, namely, the anticommutator

$$\begin{aligned}
\{C_{l,m}, C_{m,n}\} &= \frac{1}{4}\{\mathbb{1} - \sum_{\mu} \sigma_l^{\mu}\sigma_m^{\mu}\,,\, \mathbb{1} - \sum_{\nu} \sigma_m^{\nu}\sigma_n^{\nu}\} \\
&= \frac{\mathbb{1}}{2} - \frac{1}{2}\sum_{\mu} \sigma_l^{\mu}\sigma_m^{\mu} - \frac{1}{2}\sum_{\nu} \sigma_m^{\nu}\sigma_n^{\nu} + \frac{1}{4}\sum_{\mu,\nu}\{\sigma_l^{\mu}\sigma_m^{\mu},\, \sigma_m^{\nu}\sigma_n^{\nu}\} \\
&= \frac{\mathbb{1}}{2} - \frac{1}{2}\sum_{\mu} \sigma_l^{\mu}\sigma_m^{\mu} - \frac{1}{2}\sum_{\nu} \sigma_m^{\nu}\sigma_n^{\nu} + \frac{1}{4}\sum_{\mu,\nu} \sigma_l^{\mu} \underbrace{\{\sigma_m^{\mu},\, \sigma_n^{\nu}\}}_{=2\delta^{\mu\nu}} \sigma_n^{\nu} \\
&= \frac{\mathbb{1}}{2} - \frac{1}{2}\sum_{\mu} \sigma_l^{\mu}\sigma_m^{\mu} - \frac{1}{2}\sum_{\mu} \sigma_m^{\mu}\sigma_n^{\mu} + \frac{1}{2}\sum_{\mu} \sigma_l^{\mu}\sigma_n^{\mu} \\
&= C_{l,n} + C_{m,n} - C_{l,n} \qquad \square
\end{aligned}$$

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**(Σ = 12P)**