Langevin equations with multiplicative noise **(7P)**

In Langevin equations (see lecture notes) one uses *white noise*, defined as a fluctuating function $\xi(\mathbf{r},t)$ on $\mathbb{R}^d \times \mathbb{R}$ with the correlations

$$\big\langle \xi(\mathbf{r},t) \big\rangle = 0\,, \qquad \big\langle \xi(\mathbf{r},t)\xi(\mathbf{r}',t') \big\rangle = \delta^d(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

For simplicity let us assume that our problem only depends on time, considering a general Langevin equation of the form

$$\dot\rho(t) \;:=\; f(\rho(t)) + g\big(\rho(t)\big)\,\xi(t) \qquad \text{with} \qquad \big\langle \xi(t)\xi(t') \big\rangle = \delta(t-t').$$

We would like to solve this problem on a computer by discretizing the time interval $t \in [0,T]$ into N bins of width $\Delta t = T/N$ between discrete time steps $t_j = j\Delta t$.

(a) Let $q_j := \int_{t_j}^{t_{j+1}} \xi(t)$ be the accumulated noise in an interval. What is $\big\langle q_j q_k \big\rangle$? (1P)

(b) If we were to set up an Euler-like update scheme of the form

$$\rho_{j+1} = \rho_j + Af(\rho_j) + Bg(\rho_j)z_j\,,$$

where the $\{z_j\}$ are uncorrelated Gaussian random numbers with unit variance, how do we have to choose the constants A and B to lowest order in Δt? (2P)

(c) Let us finally turn to the simple case $f = 0, g = id$, called multiplicative noise[1]. In this case the Langevin equation reads $\dot\rho(t) = \rho(t)\xi(t)$. Show that a Cole-Hopf transformation $\rho(t) = e^{\Phi(t)}$ converts this problem into a Langevin equation with ordinary (non-multiplicative) noise. Guess the probability distribution $P[\Phi(t)]$. (2P)

(d) Find the corresponding probability distribution of $\rho(t)$ and determine the first and the second moment of this distribution as a function of time. (2P)

(a) Simply insert the definition of the noise correlator: (1P)

$$\langle q_j q_k \rangle \;=\; \Big\langle \int_{t_j}^{t_{j+1}} \mathrm{d}t \int_{t_k}^{t_{k+1}} \mathrm{d}t'\, \xi(t)\xi(t') \Big\rangle \;=\; \int_{t_j}^{t_{j+1}} \mathrm{d}t \int_{t_k}^{t_{k+1}} \mathrm{d}t'\, \langle \xi(t)\xi(t') \rangle$$

$$= \int_{t_j}^{t_{j+1}} \mathrm{d}t \int_{t_k}^{t_{k+1}} \mathrm{d}t'\, \delta(t-t') \;=\; \delta_{jk} \int_{t_j}^{t_{j+1}} \mathrm{d}j = \Delta t\, \delta_{jk}\,.$$

(b) Without noise the Euler scheme is just $\rho_{j+1} - \rho_j \approx \dot\rho(t_j)\Delta t$, hence $A = \Delta t$. As for the noise amplitude B, the resulting variance of the noise in a single update, B^2, should be equal to the variance of the noise function integrated over a single bin (see (a)), which is equal to Δt. This means that $B = \sqrt{\Delta t}$. This establishes an important rule of the thumb: If a non-fluctuating quantity scale like some measure, then a noise is likely to scale as the square root of this measure. (2P)

(c) The Cole-Hopf transformation maps the relevant quantities as follows: (1P)

$$\rho(t) = e^{\Phi(t)} \quad \Rightarrow \quad \dot\rho(t) = \rho(t)\dot\Phi(t) \quad \Rightarrow \quad \dot\Phi(t) = \xi(t)$$

[1] Note that this differs from DP where we have squareroot-multiplicative noise

In the variable Φ this is basically a random walk, producing a broadening Gaussian distribution of the form (1P)

$$P[\Phi(t)] = \frac{1}{\sqrt{2\pi t}} e^{-\Phi^2(t)/(2t)}$$

(d) To this end we have to know how probability densities have to be transformed (1P)

$$P[\rho(t)] = \frac{P[\Phi(t)]}{\mathrm{d}\rho/\,\mathrm{d}\Phi} = \frac{P[\Phi(t)]}{e^{\Phi}} = \frac{P[\Phi(\rho)]}{\rho} = \frac{e^{-(\ln\rho(t))^2/(2t)}}{\sqrt{2\pi t}\,\rho(t)}$$

With *Mathematica*® we compute the first two moments, obtaining (1P)

$$\langle\rho\rangle = \int_{-\infty}^{+\infty} \mathrm{d}\rho\, P[\rho]\rho = e^{t/2}$$

$$\langle\rho^2\rangle = \int_{-\infty}^{+\infty} \mathrm{d}\rho\, P[\rho]\rho^2 = e^{2t}$$

Surprisingly the variance of this non-negative random quantity grows faster than its mean, indicating an intermittent behavior with strong irregular excursions.