EXERCISE 7.1: THE FIRST GAP (6P)

The Hamiltonian of the Ising quantum chain is defined by

$$\mathbf{H}_N = \sum_{n=0}^{N-1} \Big(\sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^z \Big)$$

where $\lambda > 0$ and $\sigma_n^{x|y|z} = \mathbb{1}_{2^n \times 2^n} \otimes \sigma^{x|y|z} \otimes \mathbb{1}_{2^{N-n-1} \times 2^{N-n-1}}$ are Pauli matrices acting at site n, assuming periodic boundary conditions. The first (or lowest) gap Δ_N is defined as the difference between the first two lowest-lying eigenvalues of $\mathbf{H}_N$.

(a) In Markov processes, the first gap of $\mathcal{L}$ is the inverse leading relaxation time. In particle physics, the first gap measures the mass. What is the interpretation of the first gap of $\mathbf{H}_N$ in the present context? (1P)

(b) With the *Mathematica*® code fragment for the tensor product $\otimes$ given in the lecture notes,[1] write a code snippet to set up the Hamiltonian for arbitrary N and λ. (2P)
Cross check: The eigenvalues for $N = 3$ (i.e. 8×8) are $\big\{-\lambda-1, -\lambda-1, \lambda-1, \lambda-1, -2\sqrt{\lambda^2-\lambda+1}+\lambda+1, 2\sqrt{\lambda^2-\lambda+1}+\lambda+1, -2\sqrt{\lambda^2+\lambda+1}-\lambda+1, 2\sqrt{\lambda^2+\lambda+1}-\lambda+1\big\}$

(c) With (b) compute *numerically*[2] the first gap of $\mathbf{H}_N$ for $N = 2, 4, 6, 8, 10, 12$ and and plot the gap for $\lambda = 0.8, 1, 1.2$ double-logarithmically as a function of N. For which λ do you get an almost straight line of points and why?[3] (2P)

(d) Compute the negative *discrete logarithmic derivative*

$$\nu(N) = -\frac{\ln[\Delta_{N+2}/\Delta_N]}{\ln[(N+2)/N]}$$

for $\lambda = 1$ and $N = 2, 4, 6, 8, 10$ and guess the value of the critical exponent ν which is defined as the limit $\nu = \lim_{N\to\infty} \nu(N)$. (1P)

⇒ **PLEASE TURN OVER**

[1] You are of course free to use any other algebraic computer systems as well.
[2] In Mathmatica: Use `N[...]` to make **H** numerical: `Eigenvalues[N[H[length,lambda]]]`
[3] The execution for $N = 12$ should take less than 20 seconds. If not, go only up to $N = 10$.

Please turn over ⇒

EXERCISE 7.2: XXZ CHAIN AND 2-POINT SCALAR OPERATORS (6P)

The XXZ-Heisenberg chain with N sites and open boundary conditions is defined by the Hamiltonian

$$H = -\sum_{j=1}^{N-1} e_j\,, \qquad e_j = -\frac{1}{2}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z - \mathbb{1}\Big)\,.$$

In this exercise we want to improve algebraic skills. Please do not construct explicit matrices via *Mathematica*® , but use commutation relations instead. Of course, you may use *Mathematica*® to verify your results.

(a) Show *algebraically* that the e_j satisfy the Temperley-Lieb algebra (c.f. 3.2) (2P)

$$e_j^2 = 2e_j\,, \qquad e_j e_{j\pm1} e_j = e_j\,, \qquad [e_i, e_j] = 0 \text{ for } |i-j| \geq 2$$

Hint: The Pauli matrices $(\sigma^x, \sigma^y, \sigma^z) = (\sigma^1, \sigma^2, \sigma^3)$ obey the multiplication rule $\sigma^\mu\sigma^\nu = \delta_{\mu\nu}\mathbb{1} + i\sum_{\tau=1}^{3}\epsilon_{\mu\nu\tau}\sigma^\tau$.

(b) Verify *algebraically* that the $SU(2)$ operators

$$S^\pm = \sum_{j=1}^{N}\sigma^\pm\,, \qquad S^z = \frac{1}{2}\sum_{j=1}^{N}\sigma^z \qquad \sigma_j^\pm = \frac{1}{2}(\sigma_j^x + i\sigma_j^y)$$

obey the commutation relations $[S^+, S^-] = 2S^z$ and that they commute with the operators $e_1, \ldots, e_{N-1}$. (2P)

(c) The two-point scalar operator is defined as

$$C_{l,m} := \frac{1}{2}\Big(\mathbb{1} - \vec{\sigma}_l \cdot \vec{\sigma}_m\Big)$$

Prove that (1P)

$$C_{l,m} = (\mathbb{1} - e_{m-1})(\mathbb{1} - e_{m-2})\cdots(\mathbb{1} - e_{l+1})\,e_l\,(\mathbb{1} - e_{l+1})(\mathbb{1} - e_{l+2})\cdots(\mathbb{1} - e_{m-1})\,,$$

(implying via (b) that the two-point operator $C_{m,n}$ is SU(2)-invariant) and show that the correlator obeys the recurrence relation ($l<m<n$) (1P)

$$C_{l,n} = C_{l,m} + C_{m,n} - C_{l,m}C_{m,n} - C_{m,n}C_{l,m}.$$

(Σ = 12P)

To be handed in electronically until Wednesday, June 10, 2020, 12:00, on WueCampus according to our Corona guidelines on the web page `cs.hayehinrichsen.de`.