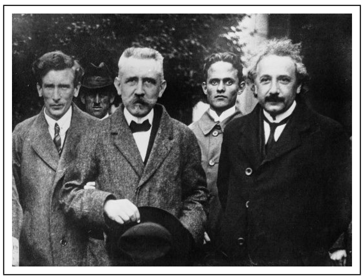

Paul Langevin and Albert Einstein 1923

## Exercise 10.1: Stochastic PDEs: Ito and Stratonovic (5P)

For the Langevin equation $\dot{y}(t) = y(t)\xi(t)$ with the initial condition $y(0) = 1$, we would like to show that the two numerical update schemes

$$\begin{aligned} \text{Ito:} \quad y_{j+1} &= y_j + \sqrt{\Delta t} y_j z_j \\ \text{Stratonovic:} \quad y_{j+1} &= y_j + \sqrt{\Delta t}\frac{1}{2}\Big(y_j + y_{j+1}\Big) z_j \end{aligned}$$

are different and how they can be related. Here $y_j \simeq y(n\Delta t)$ is a discretization with $y_0 = 1$ while the $z_j$ are uncorrelated Gaussian random numbers with unit variance,

(a) Show that in the Ito scheme the expectation value is $\langle y_j \rangle = 1$ for all $j$, meaning that the Ito scheme yields **wrong** results. (1P)

(b) Solve the Stratonovic update rule for $y_{j+1}$ and argue why this update prescription could be numerically unstable. (1P)

(c) Let us ignore the instability problem discovered in (b) and let us Taylor-expand the right-hand side of the update rule obtained in (b) up to fourth order in $\sqrt{\Delta t}$ around $\Delta t = 0$. Show that in the limit $\Delta t \to 0$ the Stratonovic update gives the correct expectation value. (2P)

(d) Add an additional deterministic drift term of the form $+\lambda \Delta t$ in the Ito scheme such that it yields the correct result in the limit $\Delta t \to 0$. (1P)

## Exercise 10.2: Schwinger-Dyson equation for DP (7P)

The Langevin equation of directed percolation (DP) reads:

$$\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = D\nabla^2\phi(\mathbf{x}, t) + a\phi(\mathbf{x}, t) - \frac{\lambda}{2}\phi(\mathbf{x}, t)^2 + \xi(\mathbf{x}, t)\,,$$

where $\xi(\mathbf{x}, t)$ is a field-dependent white noise with the correlations

$$\langle \xi(\mathbf{x}, t) \rangle = 0\,, \qquad \langle \xi(\mathbf{x}, t)\xi(\mathbf{x}', t') \rangle = 2\Gamma\phi(\mathbf{x}, t)\delta^d(\mathbf{x} - \mathbf{x}')\delta(t - t').$$

The parameters $D, a, \lambda, \Gamma$ are given constants. The parameter $a$ is the critical parameter which has to be tuned to a particular value at the transition

*Please turn over* ⇒

(a) Determine the parameter $a$, the exponents $z, \chi$, and the dimension $d$ in such a way that the Langevin equation is invariant under scale transformations (1P)

$$x \to x' = bx\,, \quad t \to t' = b^z t\,, \quad \phi(\mathbf{x},t) \to \phi'(\mathbf{x}',t') = b^\chi \phi(\mathbf{x},t)\,.$$

(b) Suppose that the parameters $D, a, \lambda, \Gamma$ have some given values. If one performs an infinitesimal scale transformation with a dilatation $b = 1+\epsilon$ $(|\epsilon| \ll 1)$, then the resulting Langevin equation can be considered as formally equivalent to the old one but with slightly different coefficients $D', a', \lambda', \Gamma'$. Determine these new coefficients as a function of the old ones to first order in $\epsilon$. (1P)

(c) Apply the Fourier transformation

$$\phi(\mathbf{k},\omega) = \int \mathrm{d}^d x\ \mathrm{d}t\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\phi(\mathbf{x},t)\,, \qquad \phi(\mathbf{x},t) = \frac{1}{(2\pi)^{d+1}}\int \mathrm{d}^d k\ \mathrm{d}\omega\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\phi(\mathbf{k},\omega)$$

to the Langevin equation to derive a transformed equation in $\phi(\mathbf{k},\omega)$. (1P)
What are the correlations of the Fourier-transformed noise $\xi(\mathbf{k},\omega)$? (1P)

(d) Considering a field theory "at the tree level" means to drop the interaction term ($\phi^2$ in our case or the corresponding convolution term in the Fourier-transformed equation). On the tree level the Langevin equation can be written as a linear differential equation on the left hand side and the noise term on the right hand side. Setting the noise term to zero, determine the bare (=tree level, denoted by an index 0) Greens function $G_0(\mathbf{k},\omega)$ and find a formal solution for $\phi_0(\mathbf{k},\omega)$. (1P)

(e) Inserting this result back into the Fourier-transformed Langevin equation show that one obtains an integral equation (1P)

$$\phi(\mathbf{k},\omega) = \phi_0(\mathbf{k},\omega) - \frac{\lambda}{2}G_0(\mathbf{k},\omega)\frac{1}{(2\pi)^{d+1}}\int \mathrm{d}^d k'\ \mathrm{d}\omega'\ \phi(k',\omega')\phi(\mathbf{k}-\mathbf{k}',\omega-\omega')$$

(f) Suggest an approximation for $\phi(\mathbf{k},\omega)$. (1P)

**($\Sigma$ = 12P)**

To be handed in electronically until Wednesday, July 01, 2020, 12:00, on WueCampus according to our Corona guidelines on the web page `cs.hayehinrichsen.de`.