# SAMPLE SOLUTIONS EXERCISE 3

## EXERCISE 3.1: SYMMETRY AND DEGENERACIES (5P)

(a) Suppose that the operators $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy the commutation relation $[\mathbf{a}, \mathbf{b}] = \mathbf{c}$. Show that the operators $\mathbf{A} = \mathbf{a} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{a}$, $\mathbf{B} = \mathbf{b} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{b}$, and $\mathbf{C} = \mathbf{c} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{c}$ obey exactly the same commutation relation $[\mathbf{A}, \mathbf{B}] = \mathbf{C}$. (1P)

(b) Assume that $\sigma^+, \sigma^-, \mathbf{m}$ obey the $su(2)$ algebra $[\sigma^+, \sigma^-] = 2\mathbf{m}$ and $[\mathbf{m}, \sigma^\pm] = \pm\sigma^\pm$. Then the operators $\mathbf{S}^\pm = \sigma^\pm \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^\pm$ and $\mathbf{M} = \mathbf{m} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{m}$ obey the same algebra. Show without using a representation that the so-called *Casimir operator*

$$\mathcal{C} = \frac{1}{2}(\mathbf{S}^+\mathbf{S}^- + \mathbf{S}^-\mathbf{S}^+) + \mathbf{M}^2$$

commutes with $\mathbf{S}^\pm$ and $\mathbf{M}$. (2P)

(c) Use the Pauli representation $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ and $m = \frac{1}{2}\sigma^z$ in order to verify that the 2-site exclusion process with symmetric rates $w_L = w_R = 1$ and $\alpha = \beta = 0$ is (up to a minus sign and an offset proportional to the identity matrix) equal to the Casimir operator $\mathcal{C}$, proving that this model is $SU(2)$-invariant. (2P)

### SAMPLE SOLUTION

(a) Here we simply apply the calculation rules for tensor products:

$$\begin{aligned}[\mathbf{A}, \mathbf{B}] &= (\mathbf{a} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{a})(\mathbf{b} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{b}) - (\mathbf{b} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{b})(\mathbf{a} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{a}) \\ &= +\mathbf{ab} \otimes \mathbb{1} + \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a} + \mathbb{1} \otimes \mathbf{ab} \\ &\quad - \mathbf{ba} \otimes \mathbb{1} - \mathbf{b} \otimes \mathbf{a} - \mathbf{a} \otimes \mathbf{b} - \mathbb{1} \otimes \mathbf{ba} \\ &= [\mathbf{a}, \mathbf{b}] \otimes \mathbb{1} + \mathbb{1} \otimes [\mathbf{a}, \mathbf{b}] \;=\; \mathbf{c} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{c} \;=\; \mathbf{C}\end{aligned}$$

(b) We first show that $[\mathcal{C}, \mathbf{S}^+] = 0$. A straight-forward algebraic calculation gives:

$$[\mathbf{M}^2, \mathbf{S}^+] = \mathbf{M}[\mathbf{M}, \mathbf{S}^+] + [\mathbf{M}, S^+]\mathbf{M} \;=\; \mathbf{S}^+\mathbf{M} + \mathbf{M}\mathbf{S}^+$$

$$\begin{aligned}\frac{1}{2}[\mathbf{S}^+\mathbf{S}^- + \mathbf{S}^-\mathbf{S}^+, \mathbf{S}^+] &= \frac{1}{2}\Big([\mathbf{S}^+\mathbf{S}^-, \mathbf{S}^+] + [\mathbf{S}^-\mathbf{S}^+, \mathbf{S}^+]\Big) \\ &= \frac{1}{2}\Big(\mathbf{S}^+ \underbrace{[\mathbf{S}^-, \mathbf{S}^+]}_{=-2\mathbf{M}} + \underbrace{[\mathbf{S}^+, \mathbf{S}^+]}_{=0} \mathbf{S}^- + \mathbf{S}^- \underbrace{[\mathbf{S}^+, \mathbf{S}^+]}_{=0} + \underbrace{[\mathbf{S}^-, \mathbf{S}^+]}_{=-2\mathbf{M}} \mathbf{S}^+\Big) \\ &= -\big(\mathbf{S}^+\mathbf{M} + \mathbf{M}\mathbf{S}^+\big)\end{aligned}$$

Adding both equations gives $[\mathcal{C}, \mathbf{S}^+] = 0$. Similarly we can prove that $[\mathcal{C}, \mathbf{S}^-] = 0$. It remains to be shown that $[\mathcal{C}, \mathbf{M}] = 0$:

$$\begin{aligned}[\mathcal{C}, \mathbf{M}] &= \frac{1}{2}[\mathbf{S}^+\mathbf{S}^- + \mathbf{S}^-\mathbf{S}^+ + \mathbf{M}^2, \mathbf{M}] \;=\; \frac{1}{2}\Big([\mathbf{S}^+\mathbf{S}^- + \mathbf{S}^-\mathbf{S}^+, \mathbf{M}] + \underbrace{[\mathbf{M}^2, \mathbf{M}]}_{=0}\Big) \\ &= \frac{1}{2}\Big(\mathbf{S}^+[\mathbf{S}^-, \mathbf{M}] + [\mathbf{S}^+, \mathbf{M}]\mathbf{S}^- + \mathbf{S}^-[\mathbf{S}^+, \mathbf{M}] + [\mathbf{S}^-, \mathbf{M}]\mathbf{S}^+\Big) \\ &= \frac{1}{2}\Big(\mathbf{S}^+\mathbf{S}^- - \mathbf{S}^+\mathbf{S}^- - \mathbf{S}^-\mathbf{S}^+ + \mathbf{S}^-\mathbf{S}^+\Big) \;=\; 0.\end{aligned}$$

(c) In the Pauli basis the 1-site operators are given by the matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{m} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding 2-site operators, which have been shown to obey the same algebra (you may check it here again) is given by

$$\mathbf{S}^+ = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{S}^- = (\mathbf{S}^+)^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is then straight-forward to compute the Casimir and to compare it with the Liouvillian:

$$\mathcal{C} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \Leftrightarrow \quad \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

hence $\mathcal{C} = 2\mathbb{1}_{4\times4} - \mathcal{L}$. For this reason, $\mathcal{L}$ commutes with $\mathbf{S}^+, \mathbf{S}^-, \mathbf{M}$, leading to $SU(2)$-like degeneracies in the spectrum, namely, one triplet and a singlet (see lecture notes).

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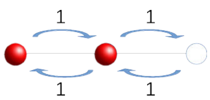
## EXERCISE 3.2: SPECTRUM-GENERATING ALGEBRA (7P)

Let us consider a three-site symmetric exclusion process ($w_L = w_R = 1$) with closed boundaries as shown in the figure. Its Liouvillian is given by $\mathcal{L} = \mathcal{L}_{12} + \mathcal{L}_{23}$ where




$$X := \mathcal{L}_{12} = \mathcal{L}^{(2)} \otimes \mathbb{1}, \quad Y := \mathcal{L}_{23} = \mathbb{1} \otimes \mathcal{L}^{(2)}.$$

$\mathcal{L}$ has the eigenvalues $\{0,0,0,0,1,1,3,3\}$ (on quartet and two doublets). Our aim is to compute the eigenvalues of $\mathcal{L}$ solely by algebraic methods without using any representation, in a similar way as one solves the harmonic oscillator in terms of $a, a^\dagger$.

(a) Verify that the matrices $X$ and $Y$ obey the so-called *Temperley-Lieb algebra* (1P)

| | | |
|---|---|---|
| *Idempotence* | $X^2 = 2X$, | $Y^2 = 2Y$ |
| *Braid group property* | $XYX = X$, | $YXY = Y$ |

Please use from now on exclusively the symbols $X$ and $Y$ and the four algebraic relations given above. Do not use their explicit matrix representation.

(b) Specify the *monomials* of the algebra, i.e., the elementary words formed by the letters $X$ and $Y$ that cannot be reduced by means of the algebraic relations. (1P)

(c) Joining two monomials by concatenation and applying the algebraic relations one obtains another monomial. List all possible results in a table. (2P)

(d) Now consider a *polynomial*, i.e., a linear combination of all monomials with arbitrary coefficients. Apply the Liouvillian to this polynomial and find the resulting polynomial, showing how the Liouvillian acts in the linear space of polynomials. (1P)

(e) Determine the *eigenpolynomials* of $\mathcal{L} = X + Y$, i.e., those polynomials which are mapped by $\mathcal{L}$ onto themselves up to a factor. Calculate the corresponding eigenvalues and compare them with the spectrum given above. (2P)

## Sample Solution

There are in principle two ways: We may include the identity in the algebra or we may not. The following sample solution describes the latter.

(a) As can be checked by using *Mathematica*® , the two matrices

$$X = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&-1&0&0&0\\ 0&0&0&1&0&-1&0&0\\ 0&0&-1&0&1&0&0&0\\ 0&0&0&-1&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0\\ 0&-1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&-1&1&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

do obey the given algebraic relations.

(b) The algebraic rules $X^2 \propto X$ and $Y^2 \propto Y$ imply that the adjacent symbols in a monomial have to be different (because otherwise you could apply those rules and reduce them). This means that monomial can only consist of alternating sequences of $X$ and $Y$. However, such sequences with three or more symbols are reduced by means of the braid group relations $XYX = X$ and $YXY = Y$. Therefore, the allowed monomials are words consisting of at most two alternating symbols, meaning that there are in total four monomials, namely, $\{X, Y, XY, YX\}$. Sometimes it is useful to add the empty work or the identity, denoted as $E$ or $\mathbb{1}$, but in the present context this does not make a difference.

(c) The map table reads:

| | X | Y | XY | YX |
|---|---|---|---|---|
| X | XX | XY | XXY | XYX |
| Y | YX | YY | YXY | YYX |
| XY | XYX | XYY | XYXY | XYYX |
| YX | YXX | YXY | YXXY | YXYX. |

Applying the algebraic relations the results can be expressed again in terms of the monomials as

| | X | Y | XY | YX |
|---|---|---|---|---|
| X | 2X | XY | 2XY | X |
| Y | YX | 2Y | Y | 2YX |
| XY | X | 2XY | XY | 2X |
| YX | 2YX | Y | 2Y | YX |

(d)

$$\begin{aligned} &(X+Y)\,(aX+bY+cXY+dYX) = \\ &aXX+bXY+cXXY+dXYX+aYX+bYY+cYXY+dYYX = \\ &2aX+bXY+2cXY+dX+aYX+2bY+cY+2dYX = \\ &\underbrace{(2a+d)}_{=a'}X+\underbrace{(2b+c)}_{=b'}Y+\underbrace{(b+2c)}_{=c'}XY+\underbrace{(a+2d)}_{=d'}YX \end{aligned}$$

(e) As can be seen, this map can be interpreted as a linear map in the 4-dimensional space of the algebra

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \mathcal{L} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

The eigenvectors of this 4×4 matrix read

$$\{\{1,0,0,1\},\{0,1,1,0\},\{-1,0,0,1\},\{0,-1,1,0\}\}$$

This implies that the corresponding eigenpolynomials are given by

$$X + YX, \quad Y + XY, \quad -X + YX, \quad -Y + XY.$$

The eigenvalues of the 4×4 matrix are $\{3,3,1,1\}$.
As can be seen, the eigenvalues of the original 8×8-Liouvillian $\{3,3,1,1,0,0,0,0\}$ have the same numerical values but different degeneracies.

**Remark**: If one adds the empty word (identity) $E$ as an additional monomial, which is represented by a unit matrix, one obtains a 5×5 eigenvalue problem. This contains also the stationary state with the eigenvalue 0 and the eigenword

$$3E - 2X - 2Y + XY + YX.$$

Including the eigenvalue zero we can see that all possible eigenvalues of the original spectrum can be reproduced algebraically, although with different degeneracies. Therefore, the Temperley-Lieb-Algebra) is also referred to as the *spectrum-generating algebra* or in short the *spectral Algebra* of the Liouville-Operator.

It is also possible to work with the Temperley-Lieb-Algebra on chains with more than three sites. To this end one associates a Temperley-Lieb operator $e_i$ to every pair of sites $i, i+1$. All these Temperley-Lieb operators are idempotent ($e_i^2 = 2e_i$) and fulfill the braid group relation with their nearest neighbors $e_i e_{i\pm1} e_i = e_i$. With non-adjacent operators, however, they commute, i.e. $[e_i, e_j] = 0$ for $|i-j| \geq 2$. For more information please refer to the lecture notes.

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**($\Sigma = 12$P)**