

PHYSICS OF COMPLEX SYSTEMS

LECTURE AND TUTORIALS – PROF. DR. HAYE HINRICHSSEN – B. SC. NILS PLÄHN – SS 2020

SAMPLE SOLUTIONS EXERCISE 2

EXERCISE 2.1: TENSOR PRODUCT \otimes (6P)

Consider two diagonalizable matrices A and B with dimensions $M \times M$ and $N \times N$, respectively. Show that

$$(a) \quad \text{Tr}[A \otimes B] = \text{Tr}[A]\text{Tr}[B]. \quad (1P)$$

$$(b) \quad \det(A \otimes B) = \det(A)^N \det(B)^M. \quad (2P)$$

$$(c) \quad \text{rank}(A \otimes B) = \text{rank}(A)\text{rank}(B). \quad (1P)$$

(d) Let $\sigma^{x,y,z}$ be the usual Pauli matrices. Compute the 4×4 matrix

$$H = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z$$

and compute the eigenvalues of H . Try to explain the degeneracies. (2P)

SAMPLE SOLUTION

(a) This is very simple:

$$\text{Tr}[A \otimes B] = \sum_{k=1}^{MN} (A \otimes B)_k = \sum_{i=1}^M \sum_{j=1}^N A_i B_j = \left(\sum_{i=1}^M A_i \right) \left(\sum_{j=1}^N B_j \right) = \text{Tr}[A]\text{Tr}[B]$$

(b) For the determinant there are several solutions. One is that we diagonalize A and B by transformations U and V such that $A' = UAU^{-1}$ and $B' = VB^{-1}V^{-1}$ is diagonal, hence

$$A' \otimes B' = (UAU^{-1}) \otimes (VBV^{-1}) = (U \otimes V)(A \otimes B)(U \otimes V)^{-1}$$

is also diagonal. This transformation does not change the determinant, meaning that $\det(A \otimes B) = \det(A' \otimes B')$. Let a_i and b_i be the diagonal elements (eigenvalues) of A' and B' . Then

$$\begin{aligned} \det(A \otimes B) &= \det(A' \otimes B') = \prod_{i=1}^M \prod_{j=1}^N (a_i b_j) = \left(\prod_{i=1}^M a_i^N \right) \left(\prod_{j=1}^N b_j^M \right) \\ &= \det(A')^N \det(B')^M = \det(A)^N \det(B)^M. \end{aligned}$$

Notice in particular the third equality: Powers and sums behave differently. If you pull out a common factor from a sum, you simply place this factor in front of the sum. If you pull out a common factor from a product, you get this factor raised to the power of the number of factors over which the product is carried out.

(c) Clearly, the image of $A \otimes B$ is given by the tensor product of the images:

$$\text{img}(A \otimes B) = \text{img}(A) \otimes \text{img}(B).$$

Since $\text{rank}(A) = \dim(\text{img}(A))$ we have

$$\begin{aligned} \text{rank}(A \otimes B) &= \dim[\text{img}(A \otimes B)] = \dim[\text{img}(A) \otimes \text{img}(B)] \\ &= \dim[\text{img}(A)] \cdot \dim[\text{img}(B)] = \text{rank}(A)\text{rank}(B) \end{aligned}$$

(d)

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

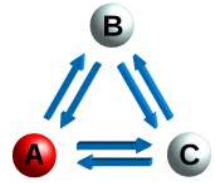
The eigenvalues are $(-3, 1, 1, 1)$. Remark: This is a singlet and a triplet, reflecting the underlying $SU(2)$ -symmetry of H .

EXERCISE 2.2: MASTER EQUATION

(6P)

Consider a Markov jump process with three possible configurations A , B , and C . Suppose that the rates for this system are given by

$$\begin{aligned} w_{A \rightarrow B} &= 3 s^{-1}, & w_{B \rightarrow A} &= 1 s^{-1}, \\ w_{A \rightarrow C} &= 2 s^{-1}, & w_{C \rightarrow A} &= 2 s^{-1}, \\ w_{B \rightarrow C} &= 1 s^{-1}, & w_{C \rightarrow B} &= 3 s^{-1}. \end{aligned}$$



- Compute the matrix of the Liouville operator \mathcal{L} in the configuration basis. (1P)
- Determine its eigenvalues as well as the left and right eigenvectors. (1P)
- Find the stationary probability distribution

$$|P_{stat}\rangle = \begin{pmatrix} P_A \\ P_B \\ P_C \end{pmatrix}$$

in the limit $t \rightarrow \infty$ and show that in this case the stationary probability currents $J_{c \rightarrow c'}^{stat} := P_c^{stat} w_{c \rightarrow c'}$ cancel pairwise, i.e., $J_{c \rightarrow c'} = J_{c' \rightarrow c}$. (1P)

- Let the system start in configuration A . Expand the initial probability distribution

$$|P_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

as a linear combination of the right eigenvectors. (2P)

- (e) With this initial configuration compute $P_A(t)$, $P_B(t)$, and $P_C(t)$ explicitly as a function of time in terms of the eigenmode decomposition. (1P)

SAMPLE SOLUTION

- (a) Suppressing the units of time, the Liouvillian in the ABC basis is given by

$$\mathcal{L} = \begin{pmatrix} 5 & -1 & -2 \\ -3 & 2 & -3 \\ -2 & -1 & 5 \end{pmatrix}.$$

- (b) The eigenvalues and left/right eigenvectors read (T denotes the transpose, a column vector):

eigenvalue	left eigenvector	right eigenvector
0	$(1, 1, 1)$	$(1, 3, 1)^T$
5	$(3, -2, 3)$	$(1, -2, 1)^T$
7	$(-1, 0, 1)$	$(-1, 0, 1)^T$

- (c) To find the stationary probability distribution the right eigenvector to the eigenvalue 0 has to be normalized such that $\langle 1|P^{stat}\rangle = 1$:

$$|P^{stat}\rangle = \begin{pmatrix} 1/5 \\ 3/5 \\ 1/5 \end{pmatrix}.$$

It is now easy to verify that the probability currents cancel one another:

$$\begin{aligned} J_{A\rightarrow B}^{stat} &= P_A^{stat} w_{A\rightarrow B} = \frac{1}{5}3 = \frac{3}{5}1 = P_B^{stat} w_{B\rightarrow A} = J_{B\rightarrow A}^{stat} \\ J_{A\rightarrow C}^{stat} &= P_A^{stat} w_{A\rightarrow C} = \frac{1}{5}2 = \frac{1}{5}2 = P_C^{stat} w_{C\rightarrow A} = J_{C\rightarrow A}^{stat} \\ J_{B\rightarrow C}^{stat} &= P_B^{stat} w_{B\rightarrow C} = \frac{3}{5}1 = \frac{1}{5}3 = P_C^{stat} w_{C\rightarrow B} = J_{C\rightarrow B}^{stat} \end{aligned}$$

Remark: This is known as the condition of *detailed balance*.

- (d) Solving the system of equations

$$a \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

we obtain $a = \frac{1}{5}$ (the normalization already computed above), $b = \frac{3}{10}$, and $c = -\frac{1}{2}$. (Note that different answers are possible since eigenvectors can be rescaled freely and even the total sign can be different.)

- (e) Suppressing the unit of time we simply have to insert the corresponding exponential functions:

$$|P(t)\rangle = \begin{pmatrix} P_A(t) \\ P_B(t) \\ P_C(t) \end{pmatrix} = a \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-5t} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-7t} = \begin{pmatrix} \frac{1}{5} + \frac{e^{-7t}}{2} + \frac{3e^{-5t}}{10} \\ \frac{3}{5} - \frac{3e^{-5t}}{5} \\ \frac{1}{5} - \frac{e^{-7t}}{2} + \frac{3e^{-5t}}{10} \end{pmatrix}$$

($\Sigma = 12P$)