SAMPLE SOLUTIONS EXERCISE 1

Warmup Exercise

EXERCISE 1.1: SIR MODEL APPLIED TO THE CORONA PANDEMIC (6P)

Consider the SIR-model (see lecture notes)

$$\dot{S}(t) = -\beta I(t)S(t)$$
$$\dot{I}(t) = +\beta I(t)S(t) - \gamma I(t)$$
$$\dot{R}(t) = +\gamma I(t).$$

with the initial conditions $S(0) = 1 - \epsilon$, $I(0) = \epsilon$, and $R(0) = 0$ where $\epsilon \ll 1$. For solving this exercise, please install the student version of *Mathematica*® from the Rechenzentrum or use another software tool of your choice.

(a) Show that the number of infected individuals increases exponentially at the beginning of the outbreak. (1P)

(b) Derive a relation between $I(t)$ and $S(t)$. (2P)
Hint: Consider the quotient $\dot{I}(t)/\dot{S}(t)$, separate variables, and integrate the resulting differential equation.

(c) Use (b) to determine the peak value of the infections I_{max}. How does the peak value depend on β? Convince yourself that *social distancing* lowers I_{max}. (1P)

(d) Solve the system of differential equations numerically with *Mathematica*® (`NDSolve`) or a similar software tool using the parameters $\beta = 0.3$, $\gamma = 0.05$, and $\epsilon = 0.01$. Plot the three quantities in a single graph as a function of time in the range $t = 0 \ldots 60$ and verify your result in (c). (2P)

SAMPLE SOLUTION

The theoretical background for understanding the SIR model can be found in the lecture notes.

(a) We consider the second equation and plug and use the fact that $S(t)$ is initially close to 1 while $I(t) \approx \epsilon$: (1P)

$$\Rightarrow \quad \dot{I}(t) \approx (\beta - \gamma)I(t) \quad \Rightarrow \quad I(t) \approx \epsilon e^{\beta - \gamma)t}.$$

(b) Consider the quotient $\dot{I}(t)/\dot{S}(t)$:

$$\dot{I}(t)/\dot{S}(t) = \frac{\frac{\mathrm{d}I}{\mathrm{d}t}}{\frac{\mathrm{d}S}{\mathrm{d}t}} = \frac{\mathrm{d}I}{\mathrm{d}S} = -1 + \frac{\gamma}{\beta S}.$$

Separate variables:

$$\mathrm{d}I = \left(-1 + \frac{\gamma}{\beta S}\right)\,\mathrm{d}S$$

Integrate both sides:

$$\int_{\epsilon}^{(I(t)} \mathrm{d}I = \int_{1-\epsilon}^{S(t)} \left(-1 + \frac{\gamma}{\beta S}\right)\,\mathrm{d}S = \int_{S(t)}^{1-\epsilon} \left(1 - \frac{\gamma}{\beta S}\right)\,\mathrm{d}S$$

The integrals can be carried out:

$$I(t) - \epsilon \;=\; 1 - S(t) - \epsilon + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{1-\epsilon}\right).$$

$$\Rightarrow \quad I(t) \;=\; 1 - S(t) + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{1-\epsilon}\right).$$

In this expression the initial infection level ϵ can be set to zero. Therefore, this part of the exercise is also correctly solve if ϵ was neglected from the beginning on.

(c) Let t_{max} be the time at which the fraction of infected individuals $I(t)$ is maximal. This means that

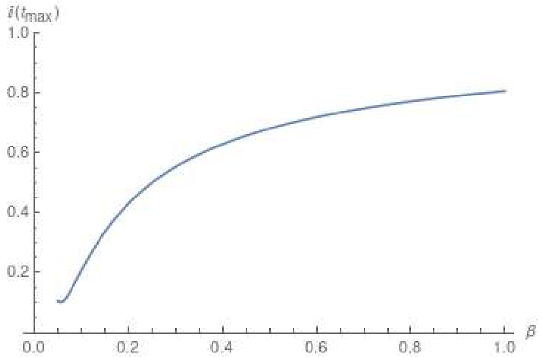
$$\dot{I}(t_{max}) = \beta I(t_{max})S(t_{max}) - \gamma I(t_{max}) \;=\; 0.$$

$$\Rightarrow \quad S(t_{max}) = \frac{\gamma}{\beta}$$

Now we can use the relation obtained in (b):

$$I(t_{max}) = 1 + \frac{\gamma}{\beta}\left[\ln\left(\frac{\gamma}{\beta(1-\epsilon)}\right)\right]$$

For $\gamma = 0.05$ this quantity as a function of β looks like this:



Social distancing means to lower the spreading exponent β and therefore lowers the peak value of the number of infections.

(d) In *Mathematica*® this can be done as follows:

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In[32]:= variables = {s[t], i[t], r[t]};
         equations = {s'[t] == -β i[t] × s[t], i'[t] == β i[t] × s[t] - γ i[t], r'[t] == γ i[t]};
         initial = {s[0] == 1 - ϵ, i[0] == ϵ, r[0] == 0};
         β = 0.3; γ = 0.05; ϵ = 0.01; T = 60;
         solution = NDSolve[Join[equations, initial], variables, {t, 0, T}];
         Plot[{s[t], i[t], r[t]} /. solution[[1]], {t, 0, T}]
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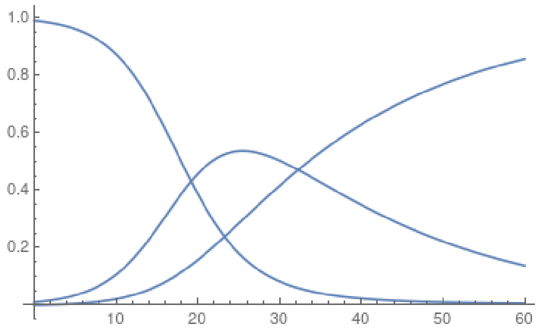

(Σ = 6P)