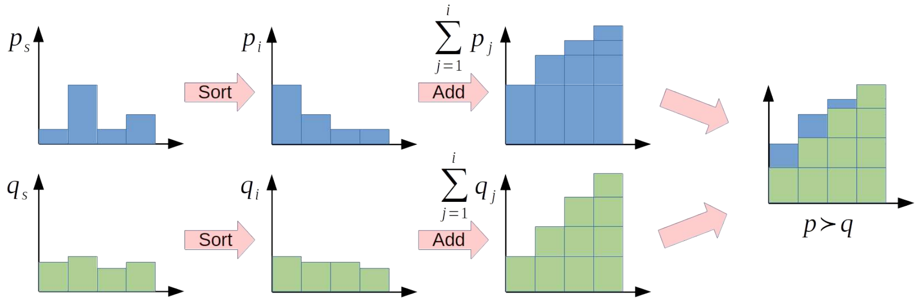

Majorization: A probability distribution p is said to *majorize* another probability distribution q if the cumulative distribution of the probabilities sorted in descending order is equal or larger.

Exercise 4.1: Maximal entropy (3P)

The Shannon entropy of a probability distribution is given by

$$H = -\sum_c p_c \ln p_c \,.$$

(a) Use Jensens inequality for convex functions to show that the Shannon entropy attains its global maximum for a uniform distribution. (1P)

(b) Prove the same statement in the framework of variational calculus. Use the method of Langrange multipliers to take the normalization constraint of the probability distribution into account. (2P)

Exercise 4.2: Majorization of probability distributions (9P)

Let $\mathbf{a} = \{a_1, a_2, \ldots, a_N\}$ and $\mathbf{b} = \{a_1, a_2, \ldots, a_N\}$ be two sets of numbers, both sorted in descending order. The set $\mathbf{a}$ is said to *majorize* the set $\mathbf{b}$ (denoted as as $\mathbf{a} \succ \mathbf{b}$) if

$$\mathbf{a} \succ \mathbf{b} \qquad \Leftrightarrow \qquad \sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} b_i \text{ for all } n = 1, \ldots, N-1 \text{ and } \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i$$

(a) Consider a Markov process in a closed system (symmetric rates). Let $\mathbf{p}(t) = \{P_s(t)\}$ be the actual set of probabilities at time t sorted in descending order. Use the master equation to prove that $\mathbf{p}(t) \succ \mathbf{p}(t')$ for $t < t'$. (2P)

(b) For given numbers $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ let

$$\begin{aligned} x_i &:= X_{i+1} - X_i && \text{for } i = 1, \ldots, N-1 \\ y_i &:= Y_i - Y_{i-1} && \text{for } i = 2, \ldots, N \\ y_1 &= Y_1 \end{aligned}$$

Prove Abel's partial sum theorem (a discrete version of partial integration): (2P)

$$\sum_{i=1}^{N} X_i y_i \;=\; X_N Y_N - \sum_{i=1}^{N-1} x_i Y_i \,.$$

Please turn over ⇒

(c) Let $f(x)$ be a concave function. Prove that $\mathbf{a} \succ \mathbf{b}$ implies that (3P)

$$\sum_{i=1}^{N} f(a_i) \;\leq\; \sum_{i=1}^{N} f(b_i)\,.$$

Hint: Apply (b) to the expression

$$\sum_{i=1}^{N} \Big(f(b_i) - f(a_i) \Big) \;=\; \sum_{i=1}^{N} \Big(\underbrace{\frac{f(b_i) - f(a_i)}{b_i - a_i}}_{=X_i} \underbrace{(b_i - a_i)}_{=y_i} \Big)$$

and try to find an inequality for *secant slopes* on concave functions.

(d) Use (c) to show that for two probability distributions with $\mathbf{p} \succ \mathbf{q}$, the Shannon entropy H satisfies the inequality $H(\mathbf{q}) \geq H(\mathbf{p})$. (1P)

Note: Together with (a), this confirms the Second Law of Thermodynamics.

(e) Show the same for the Rényi entropy H_α. (1P)

(Σ = 12P)

To be handed in electronically on Wednesday, May 20, 2020, on WueCampus according to our Corona guidelines on the web page `cs.hayehinrichsen.de`.